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Geometry of G/P-II
[The work of De Concini and Procesi and the basic conjectures]

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1. Introduction

The main aim of this paper is to show how the work of De Concini and Procesi [5] on classical invariant theory can be interpreted to suggest a generalisation of the Hodge-Young theory (*cf.* Hodge [11]; Hodge and Pedoe [12]) of standard monomials. This generalisation is given as a set of conjectures* (Conjectures I and II) in section 6. On the other hand, we also show that the results of De Concini and Procesi [5] follow as consequences of this generalisation of the Hodge-Young theory (*cf.* section 7).

Let $G=SL(n)$ and P a maximal parabolic subgroup in G so that G/P is the Grassmannian of r-dimensional vector subspaces of an n-dimensional vector space. Let f be a *lowest weight vector* in $H^0(G/P, L)$, where L denotes the ample generator of Pic G/P. If $W(G)$ denotes the Weyl group of G, one sees that the subgroup of $W(G)$ which fixes the one-dimensional linear space spanned by f, is precisely the Weyl group $W(P)$ of P. The translates of f by $W(G)$ can therefore be indexed by $W(G)/W(P)$ and we set

$$p_\tau = \tau . f,\ \tau \in W(G)/W(P).$$

Let us call a *standard monomial of length m on G/P*, an expression of the following form:

$$(*) \qquad p_{\tau_1} p_{\tau_2} \cdots p_{\tau_m};\ \tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_m$$

where by $\tau_1 \geqslant \tau_2$ etc. we take the canonical partial order in $W(G)/W(P)$. By the Hodge-Young theory we mean the theorem which states that distinct standard monomials of length m on G/P form a basis of $H^0(G/P, L^m)$, as well as its generalisation to a Schubert variety in G/P (*cf.* section 2). Let us call an expression of the following form

$$(\tau_1, \ldots, \tau_m),\ \tau_i \in W(G)/W(P),\ \tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m$$

*These conjectures have now been proved in collaboration with C. Musili.

a *Young diagram of length m* in $W(G)/W(P)$. We have a canonical identification $W(G)/W(P) \approx I_n(r)$, where

$$I_n(r) = \{(i_1, \ldots, i_r)/1 \leqslant i_1 < i_2 \ldots < i_r \leqslant n\}.$$

One knows that if $\tau_1=(i_1, \ldots, i_r)=(i)$ and $\tau_2=(j_1, \ldots, j_r)=(j)$, then

$$\tau_1 \geqslant \tau_2 \text{ in } W(G)/W(P) \Longleftrightarrow i_k \geqslant j_k,\ 1 \leqslant k \leqslant r.$$

Thus Young diagrams in $W(G)/W(P)$ are just certain types of Young diagrams taken in the usual sense (Weyl [23]) and the Hodge-Young theory gives a canonical identification of the set of Young diagrams of length m in $W(G)/W(P)$ with a basis of $H^\circ(G/P, L^m)$. One can define more generally what could be called *Young diagrams in* $W(G)$ (these are just the Young diagrams in the general sense as is usually understood (Weyl [23])), by which we can similarly write a basis of $H^\circ(G/B, M)$, M being an arbitrary line bundle on the *flag variety* G/B of G (Hodge [11]).

It is natural to ask whether the above theory could be carried over to the general case of a semi-simple algebraic group G and P, a parabolic subgroup of G. In (Seshadri [22]) such a generalisation has been done for the case when P is *minuscule* i.e. it is a maximal parabolic subgroup whose associated fundamental weight ω is *minuscule*. Recall Bourbaki [2], Seshadri, [22] that a fundamental weight ω is called *minuscule* if in the irreducible representation V of the group G', with highest weight ω, G' being the semisimple group defined over $\mathbf{C}$ with the same root system as G, *all the weights of* V are translates of ω by the Weyl group $W(G')$ of G'. Let us define $\{p_\tau\}$, $\tau \in W(G)/W(P)$, as well as standard monomials in $\{p_\tau\}$ as above. Then it has been shown (Seshadri [22]) that exactly the same results as above (stated for the case $G=SL(n)$) hold good. This generalisation is, however, not satisfactory enough, since when G is (almost) simple and not of type **A**, there are not many minuscule weights and there are exceptional G with no minuscule weights at all (Bourbaki [2]; Seshadri [22]).

Let G be a semi-simple algebraic group and P a maximal parabolic subgroup of G. Let $X(\tau)$ denote the Schubert variety associated to $\tau \in W(G)/W(P)$. We have

$$\tau_1 \geqslant \tau_2 \text{ in } W(G)/W(P) \Longleftrightarrow X(\tau_1) \supseteq X(\tau_2).$$

Let $[X(\tau)]$ denote the class determined by $X(\tau)$ in the Chow ring $\mathrm{Ch}(G/P)$ of G/P. Let H denote the Schubert variety of codimension one in G/P. It can be shown that

$$[X(\tau)] \cdot [H] = \sum_{Y \in S_\tau} d_Y [Y],\ d_Y \geqslant 0$$

where centre dot denotes multiplication in $\mathrm{Ch}(G/P)$ and S_τ denotes the set of Schubert varieties of codimension one in $X(\tau)$. We call d_Y the *intersection multiplicity* of Y in $[X(\tau)] \cdot [H]$. If P is minuscule, it can be shown that $d_Y = 1$. Let us call P to be of *classical type* (see section 6) if $d_Y \leqslant 2$ for any pair of Schubert varieties $X(\tau)$ and Y as above. It can be shown that for any G there is always a maximal parabolic subgroup P which is of classical type and that if G is a classical group every maximal parabolic subgroup is of classical type. Let us suppose that P is of classical type.

Let us call a pair of elements (τ, φ) in $W(G)/W(P)$ to be an *admissible pair* in $W(G)/W(P)$ if either $\tau = \varphi$ (in which case it is called a *trivial* admissible pair) or $\tau \neq \varphi$ (in which case it is called a *non-trivial admissible pair*) and there exist $\{\tau_i\}$, $1 \leqslant i \leqslant r$, $\tau_i \in W(G)/W(P)$ such that

(i) $\tau_1 = \tau$, $\tau_m = \varphi$; $\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m$

(ii) $X(\tau_i)$ is of codimension one in $X(\tau_{i-1})$, $2 \leqslant i \leqslant m$.

(iii) the intersection multiplicity of $X(\tau_i)$ in $[X(\tau_{i-1})] \cdot [H]$ is 2, $2 \leqslant i \leqslant m$.

Let (τ_1, φ_1), (τ_2, φ_2) be two admissible pairs in $W(G)/W(P)$. We define

$$(\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2) \Longleftrightarrow \varphi_1 \geqslant \tau_2.$$

We call a *Young diagram of length m in* $W(G)/W(P)$, a sequence of m admissible pairs in $W(G)/W(P)$ of the following type:

$$(\tau_1, \varphi_1), (\tau_2, \varphi_2), \ldots, (\tau_m, \varphi_m) \text{ with } (\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2) \geqslant \ldots \geqslant (\tau_m, \varphi_m)$$

where (τ_i, φ_i), for $1 \leqslant i \leqslant m$, are admissible pairs in $W(G)/W(P)$.

Then Conjectures I state (see section 6 for the precise form) that there is a canonical identification of the set of Young diagrams of length m in $W(G)/W(P)$ with a basis of $H^\circ(G/P, L^m)$ ($L =$ the ample generator of Pic G/P); in fact if we denote by $p_{(\tau,\varphi)}$ the element in $H^\circ(G/P, L)$, associated to an admissible pair (τ, φ) in $W(G)/W(P)$, then the canonical basis in $H^\circ(G/P, L^m)$ is given by *standard monomials in* $p_{(\tau,\varphi)}$, of length m, defined in a similar manner as above. In Conjectures II, we take G to be a *classical group*. We then define *Young diagrams in* $W(G)$ and state a canonical identification of Young diagrams of a certain type with a basis of $H^\circ(G/B, M)$ where M is *any line bundle* on G/B ($B =$ a Borel subgroup in G).

The importance of this conjectural standard monomial theory (apart from its apparent independent interest) stems from the fact that it would provide a systematic approach towards the proof of the Kodaira type of vanishing theorems for line bundles (in the dominant chamber) on G/B and its Schubert subvarieties (Hochster [9]; Musili [18]; Lakshmibai *et al* [15]; Kempf [13]), the best theorem to date in this connection being due to Kempf [13]. In collaboration with Musili [16], we have proved the Conjectures I in the important particular case when the *fundamental weight* ω *associated to* P *is quasi-minuscule* (*equivalently a distinguished weight in the sense of* Kempf [13]) and this case suffices to deduce the results of Kempf [13]. (We call a fundamental weight ω *quasi-minuscule*, if in the irreducible representation V of the group G', with highest weight ω (G' being the semisimple group defined over $\mathbf{C}$ and having the same root system as that of G), *all the non-zero weights* are translates of ω by the Weyl group $W(G')$ of G').

For classical groups, Conjectures I have been checked for the case $m = 1$.

The connection between the above conjectures and the work of De Concini and Procesi [5] is as follows. Set:

Case (i): $X = \underbrace{W \oplus \ldots \oplus W}_{m \text{ times}} \oplus \underbrace{W^* \oplus \ldots \oplus W^*}_{m \text{ times}}$, $G = GL(W)$ and $\dim W = 2n$

Case (ii): $X = W \oplus \ldots \oplus W$ (m times), $\dim W = 2n$,
$G =$ the orthogonal group $\mathbf{O}(2n)$, $\mathbf{O}(2n) \subset GL(W)$.

Case (iii): $X = W \oplus \ldots \oplus W$ (m times), $\dim W = 2n$,
$G =$ the symplectic group $Sp(2n)$, $Sp(2n) \subset GL(W)$.

We now take the diagonal action of G on X. Set

$$X = \text{Spec } R, \; Y = \text{Spec } R^G,$$

De Concini and Procesi [5] have given a basis of the invariant subring R^G of R by means of certain types of what they call *standard monomials* and have shown that (also Hochster and Eagon [10]) for the linear case and Kutz [14] for the orthogonal case)

$$Y = \begin{cases} \text{determinantal variety in the space } M_m \text{ of } (m \times m) \text{ matrices in case (i)} \\ \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{Sym } M_m \text{ of } (m \times m) \text{ symmetric matrices in case (ii)} \\ \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{Sk } M_m \text{ of } (m \times m) \text{ skew-symmetric matrices in case (iii).} \end{cases}$$

One knows that a determinantal variety in M_m can be identified as an open subset—in fact as the *opposite big cell*—of a Schubert variety in a Grassmannian (*cf* section 2 below and Hochster [9]; Musili [19]). Then it is easily checked that the standard monomials written down by De Concini and Procesi for Case (i) are precisely the restrictions (to the opposite big cell) of the standard monomials on Grassmannians in the Hodge-Young sense as we described above. One then observes (*cf.* section 4) that a determinantal variety in Sym M_m (resp. Sk M_m) is precisely the opposite big cell of a Schubert variety in $Sp(2n)/Q$ (resp. $SO(2n)/Q$), where Q is the maximal parabolic subgroup associated to a right end root (in the Dynkin diagram, Bourbaki [2]). In the case of $SO(2n)$, it is known that Q is *minuscule* and one finds (ch. Th. 5.1) that the standard monomials of De Concini and Procesi for Case (iii) are again the restrictions to the opposite big cell of standard monomials in the sense of (Seshadri [22]). In the case of $Sp(2n)$, Q is not *minuscule* and the question arose whether the standard monomials of De Concini and Procesi for Case (ii), could be properly interpreted to suggest a good definition of standard monomials for the nonminuscule case. Such an interpretation is given in section 5 and has been the principal motivation behind Conjectures I.

The best way to read this paper is perhaps to go through section 2 first, then the statement of Theorem 4.1 and thereafter sections 5, 6 and 7. The rest of the paper (which is fairly long) is to be referred to, whenever found necessary; probably this

material could be written in a more condensed manner so as to reduce the length of the paper.

The second named author is thankful to C. Procesi for the discussions he had with him on his work with De Concini during his stay in Rome in April 1976. The project of this paper took shape during these discussions. We are thankful to N Soundararajan of the Computer group, TIFR for his crucial help in the verification of the above conjectures in low ranks; in fact he has programmed the verifications of Conjectures I for $G = Sp(2n)$, $m = 2$ as well as Conjectures II for $G = Sp(2n)$, $|m| = 2$; in particular, for these cases Conjectures I and II have been verfied for $n \leqslant 7$. We are also thankful to C. Musili for his careful reading of this paper.

1. Basic facts on $SL(n)$

We work with an algebraically closed base field K.

Let V be an m-dimensional vector space over K. We fix a basis $e_1, \ldots, e_m$ of V. We write the elements of V as column vectors of length m with entries in K, so that

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ \ldots \ e_i = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ 1 \text{ in the } i\text{th place, } 0 \text{ elsewhere.}$$

Then multiplication of elements of V by elements of $GL(m)$ on the left, makes V a (left) $GL(m)$ module. We refer to this as the *canonical $GL(m)$ module structure on V*. We set

$H = SL(m)$
$B_m =$ the Borel subgroup in $GL(m)$ formed by upper triangular matrices
$T_m =$ maximal torus in $GL(m)$ formed by diagonal matrices
$B(H) =$ the Borel subgroup $B_m \cap SL(m)$ in H
$T(H) =$ the maximal torus $T_m \cap SL(m)$ in H
$L_m =$ the Lie algebra of $GL(m)$ identified with the set of all $(m \times m)$matrices
Lie $H =$ the Lie algebra of H identified with the set of $(m \times m)$ matrices of trace zero
$N(T_m) =$ normaliser of T_m in $GL(m)$
$N(T(H)) = N(T_m) \cap H$ (it is the normaliser of $T(H)$ in H)
$W(H) = W_m = N(T(H))/T(H) = N(T_m)/T_m$ (Weyl group of H or $GL(m)$)
$S_m =$ symmetric group on the m letters $(1, \ldots, m)$

If θ is an element of S_m such that $\theta(i) = a_i$, as is customary, we write

$$\theta = \begin{pmatrix} 1 \ \ldots \ m \\ a_1 \ldots a_m \end{pmatrix} \text{ or simply } \theta = (a_1, \ldots, a_m).$$

If $M \in N(T_m)$, we check that there is a uniquely determined element $\theta = (a_1, \ldots, a_m) \in S_m$ such that in the ith column of M, the only non-zero element is in the a_ith row. We then write $M = M(\theta)$. The mapping $M \longmapsto \theta$ gives an isomorphism of $W(H)$ or W_m onto S_m, which we call the canonical isomorphism of $W(H)$ or W_m onto S_m. If $t \in T_m$ is of the form

$$(1) \qquad t = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{pmatrix}$$

we see that

$$(2) \qquad M(\theta)t\,(M(\theta))^{-1} = \begin{pmatrix} t_{b_1} & & 0 \\ & \ddots & \\ 0 & & t_{b_m} \end{pmatrix}, \quad (M(\theta))^{-1}tM(\theta) = \begin{pmatrix} t_{a_1} & & 0 \\ & \ddots & \\ 0 & & t_{a_m} \end{pmatrix}.$$

where the element $(b_1, \ldots, b_m)$ is the inverse θ^{-1} of θ in S_m.

We set:

$$X_0(T_m) \text{ (resp. } X_0(T(H))) = \text{Hom } (T_m, \mathbf{G}_m) \text{ (resp. Hom } (T(H), \mathbf{G}_m))$$

$$X(T_m) \text{ (resp. } (X\,T(H))) = X_0(T_m) \otimes_{\mathbf{Z}} \mathbf{R} \text{ (resp. } X_0(T(H)) \otimes_{\mathbf{Z}} \mathbf{R}).$$

The elements $\epsilon_i \in X_0\,(T_m)$, $1 \leqslant i \leqslant m$, defined by

$$\epsilon_i(t) = t_i,\ t \in T_m \text{ as in (1) above}$$

form a $\mathbf{Z}$-basis of $X_0(T_m)$. We denote the canonical images of ϵ_i in $X(T_m)$ by the same ϵ_i so that $\{\epsilon_i\}$, $1 \leqslant i \leqslant m$, is a basis of the $\mathbf{R}$-vector space $X(T_m)$. The canonical action of $W_m \approx S_m$ on $X_0(T_m)$ (and hence on $X(T_m)$) is defined as follows: for $\theta = (a_1, \ldots, a_m) \in S_m$ and $\chi \in X_0(T_m)$ ($\chi : T \to \mathbf{G}_m$), we define $\theta \cdot \chi$ by

$$(\theta \cdot \chi)\,(t) = \chi M(\theta)^{-1}\,tM(\theta),\ t \in T_m \text{ and } M(\theta) \in N(T_m)$$

as above. Then by (2) above, we see that

$$\theta(\epsilon_i) = \epsilon_{a_i},\ 1 \leqslant i \leqslant m.$$

We have a canonical surjective linear map

$$\varphi : X(T_m) \to X(T(H)).$$

We see that the restriction of φ to the linear subspace spanned by $\epsilon_i - \epsilon_j$ is an isomorphism and *we identify* $X(T(H))$ *with this subspace.* One knows that the set

$$R(H)=\{\epsilon_i-\epsilon_j/i\neq j\}$$

can be identified with the set of *roots* of H with respect to $T(H)$ and that

$$R^+(H)=\{\epsilon_i-\epsilon_j/i<j\}$$

is the set of *positive roots* (with respect to $B(H)$). If $\alpha=\epsilon_i-\epsilon_j \in R(H)$, we denote by U_α the unipotent subgroup ($\approx \mathbf{G}_a$) of $GL(m)$ or H defined by:

$U_\alpha=\{I+g/I=(m\times m)$ identity matrix, g an $(m\times m)$ matrix whose only non-zero entry is in the (i,j)th place$\}$.

The action of $T(H)$ on $U_\alpha(\approx \mathbf{G}_a)$ is given by the element of $X_0T(H)$, canonically associated to the root $\alpha=\epsilon_i-\epsilon_j$, since we have

$$t(g_{ij})t^{-1}=(g_{ij}t_it_j^{-1}),\ t \text{ as in (1)},\ (g_{ij}) \in GL(m).$$

The $\{U_\alpha\}$, $\alpha>0$, generate the unipotent subgroup $B^u(H)$ of $B(H)$. We see also that the canonical action of $W(H)$ $(=W_m=S_m)$ on $R(H)$ is explicitly given as follows:

$$\theta=(a_1, \ldots, a_m) \in S_m,\ \theta(\epsilon_i-\epsilon_j),=\epsilon_{a_i}-\epsilon_{a_j}.$$

The K-vector space V has a canonical (left) H-module structure induced from the canonical $GL(m)$ module structure on V. Set $l=m-1$, $m\geqslant 2$. Then $\overset{r}{\Lambda}\, V$, $1\leqslant r\leqslant l$ have canonical H-module structures. One knows that these are the *fundamental representations* of H. We see that the vector $e_1\Lambda \ldots \Lambda e_r \in \overset{r}{\Lambda}\, V$ is a highest weight vector with respect to $T(H)$ and $B(H)$ since the 1-dimensional space spanned by $e_1\Lambda \ldots \Lambda e_r$ is stable under $B(H)$. If we denote by $\{\alpha_i\}$, $1\leqslant i\leqslant l$, the simple roots $\alpha_i=\epsilon_i-\epsilon_{i+1}$ $1\leqslant i\leqslant l$ in $R(H)$ and by $\{\omega_i\}$, $1\leqslant i\leqslant l$ the fundamental weights, having the property, $\langle \omega_i, \alpha_j{}^\vee\rangle = \delta_{ij}$ (Kronecker delta) (notations as in Bourbaki [2]) we find that the weight of $e_1\Lambda \ldots \Lambda e_r$ is ω_r (we see that the weight of $e_1\Lambda \ldots \Lambda e_r$ under T_m is $\epsilon_1+\ldots+\epsilon_r-(r/m)\left(\Sigma_{i=1}^m \epsilon_i\right)$ which is precisely ω_r, see Bourbaki [2]). Thus the highest weight of the H-module $\overset{r}{\Lambda}\, V$ is ω_r.

Let $\mathbf{P}(\overset{r}{\Lambda}\, V)$ denote the projective space of 1-dimensional subspaces of $\overset{r}{\Lambda}\, V$ (in Grothendieck's sense, this should be denoted by $\mathbf{P}((\overset{r}{\Lambda}\, V)^\vee)$, $(\overset{r}{\Lambda}\, V)^\vee$=dual of $(\overset{r}{\Lambda}\, V)$. Then H operates on $\mathbf{P}(\overset{r}{\Lambda}\, V)$. We denote by P_r the isotropy subgroup of H at the point of $\mathbf{P}(\overset{r}{\Lambda}\, V)$ corresponding to $e_1\Lambda \ldots \Lambda e_r$. Then P_r is a maximal *parabolic subgroup* in H, $P_r \supset B(H)$ and H/P_r can be canonically identified with the Grassmannian of r dimensional linear subspaces in V. We see that P_r is of the form

$$\left(\begin{array}{c|c} (r\times r) & * \\ \hline 0 & (m-r)\times(m-r) \end{array}\right)$$

In fact P_r is the *maximal parabolic subgroup in H canonically associated to the simple root* α_r (Borel [1]; Lakshmibai *et al* [15]) i.e., if M_r is the *reductive part of* P_r, then M_r is generated by T_m and all U_α such that the root α is spanned by the simple roots $\{\alpha_i\}$, $i \neq r$(i.e. $\alpha_i \neq \alpha_r$) (or that the unipotent part of the radical of P_r is generated by all U_α, such that $\alpha > 0$ and $\alpha = \Sigma n_i \alpha_i$ with $n_r \neq 0$).

Let $W_r = W_{P_r}$ be the Weyl group of P_r. One knows that W_r is the isotropy subgroup of $W(H)$ at $e_1 \Lambda \ldots \Lambda e_r$ (in the sense that $\theta \in W(H) = S_m$ is in W_r if and only if $M(\theta)$ leaves stable the 1-dimensional subspace through $e_1 \Lambda \ldots \Lambda e_r$). Note that if $\theta = (a_1, \ldots, a_m)$, then

$$M(\theta) e_i = m_{a_i, i}\, e_{a_i},\ M(\theta) = (m_{ij}).$$

This implies that

$$M(\theta)(e_1 \Lambda \ldots \Lambda e_r) = \lambda e_{a_1} \Lambda \ldots \Lambda e_{a_r},\ \lambda \neq 0. \tag{3}$$

Let $I_m(r)$ denote the following set:

$$I_m(r) = \{(i_1, \ldots, i_r) / 1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant m\}.$$

Then from (3) above, we see that we have a canonical identification of $W(H)/W_r$ with $I_m(r)$; in fact, if $\theta = (a_1, \ldots, a_m) \in S_m$, then the canonical image of θ in $I_m(r)$ is obtained by taking the first r elements $a_1, \ldots, a_r$ of θ and arranging them in the increasing order. We have a canonical *partial order in* $W(H)$ (resp. $W(H)/W_r$). Given w_1, w_2 in $W(H)$ (resp. $W(H)/W_r$), this partial order can be defined by saying that given a *reduced decomposition* of w_2 (with respect to the simple reflections s_i in $W(H)$, s_i being the reflection with respect to the simple root α_i), it contains some reduced decomposition of w_1 as a 'subword' or equivalently as follows: Let $X(w_i, H/B(H))$ (resp. $X(w_i, H/P_r)$) denote the *Schubert variety* in $H/B(H)$ (resp. H/P_r) associated to w_i i.e. this variety is the Zariski closure of the *Schubert cell* $B(H) w_i e_{B(H)}$ (resp. $B(H) w_i e_{P_r}$) in $H/B(H)$ (resp. H/P_r), where we denote by $e_{B(H)}$ (resp. e_{P_r}), the point in $H/B(H)$ (resp. H/P_r) represented by the class $B(H)$ (resp. P_r) (note also that when we write $w_i\, e_{B(H)}$ (resp. $w_i\, e_{P_r}$), we use a representative of w_i in $N(T(H))$ and that $w_i\, e_{B(H)} \in H/B(H)$ (resp. $w_i\, e_{P_r} \in H/P_r$) is independent of the choice of a representative). Then

$$w_1 \leqslant w_2 \text{ in } W(H) \text{ (resp. } W(H)/W_r) \Longleftrightarrow X(w_1, H/B(H)) \subseteq (Xw_2, H/B(H))$$
$$\text{(resp. } X(w_1, H/P_r) \subseteq X(w_2, H/P_r).$$

We have a *canonical partial order* in $I_m(r)$, namely, if $(i) = (i_1, \ldots, i_r)$ and $(j) = (j_1, \ldots, j_r)$ are in $I_m(r)$, we write

$$(i) \leqslant (j) \Longleftrightarrow i_1 \leqslant j_1, \ldots, i_r \leqslant j_r.$$

Then we have the following:

(i) The canonical identification of $W(H)/W(P_r)$ with $I_m(r)$ preserves the canonical partial orders in each.

(ii) If $w \in W(H)/W(P_r)$ and $w=(i_1,\ldots,i_r)\in I_m(r)$ (identifying $W(H)/W(P_r)$ with $I_m(r)$), then

$$\dim X(w, H/P_r) = \sum_{k=1}^{r} i_k - \frac{r(r+1)}{2}$$

(iii) Let $w = (i_1, \ldots, i_r) \in I_m(r)$. Let $(j_1, \ldots, j_s)$ be the complement of $(i_1, \ldots, i_r)$ in $(1, 2, \ldots, m)$ arranged in the increasing order. Then $w_1 = (i_1, \ldots, i_r, j_1, \ldots, j_s)$ is the *minimal representative* in $W(H)$ of $w \in W(H)/W(P_r)$, in the sense that among all the representatives of w in $W(H)$, w_1 is of minimum length in $W(H)$ (see Bourbaki [2] for the definition of length).

(iv) For $w \in W(H)$, one knows that the dimension of the Schubert variety $X(w, H/B(H))$ is equal to the *length of w in* $W(H)$ denoted $l(w, W(H))$. For $w \in W(H)/W(P_r)$, we call the length of w in $W(H)/W(P_r)$, denoted $l(w, W(H)/W(P_r)$ the length in $W(H)$ of the minimal representative of w in $W(H)$. One knows that the dimension of $X(w, H/P_r)$, $w \in W(H)/W(P_r)$ is equal to $l(w, (H)/W(P_r))$.

Let more generally G be a semi-simple group (defined over K), B a Borel subgroup in G and P a parabolic subgroup in G, $P \supset B$. We denote by $W(G)$ (resp. $W(P)$) the Weyl group of G (resp. P). For $w \in W(G)$ (resp. $W(G)/W(P)$), we denote by $X(w, G/B)$ (resp. $X(w, G/P)$), the Schubert variety in G/B (resp. G/P) associated to w. Let $w \in W(G)/W(P)$; then we call an element $w_1 \in W(G)$ a *minimal* (resp. *maximal*) representative of w if (i) w_1 represents w and (ii) if w_2 is any representative of w in $W(G)$, then $l(w_2) \geqslant l(w_1)$ (resp. $l(w_2) \leqslant l(w_1)$). A minimal representative w_1 of w is characterized by the following property (Bourbaki [2]).

$$l(w_1\lambda, W(G)) = l(w_1, W(G)) + l(\lambda, W(G)), \lambda \in W(P).$$

The following can be checked easily (and we require them later):

(i) the minimal and maximal representatives in $W(G)$ of $w \in W(G)/W(P)$ are uniquely determined. Let w_i, $i = 1, 2$ be respectively the minimal and maximal representatives in $W(G)$ of w. Then we have:

(ii) $X(w_2, G/B)$ is the inverse image of $X(w, G/P)$ under the canonical morphism $G/B \to G/P$. Further, under this morphism, $X(w_1, G/B)$ maps on to $X(w, G/P)$ and the morphism $X(w_1, G/B) \to X(w, G/P)$ is birational.

(iii) $w_2 = w_1 \cdot w_0(P)$, where $w_0(P)$ is the element of maximal length in $W(P)$.

(iv) Let $w' \in W(G)/W(P)$ and w'_1, w'_2 be respectively the minimal and maximal representatives in $W(G)$ of w'. Then

$$w \leqslant w' \text{ in } W(G)/W(P) \iff w_1 \leqslant w'_1 \text{ in } W(G) \iff w_2 \leqslant w'_2 \text{ in } W(G).$$

(v) if w'_1, w'_2 are two representatives in $W(G)$ of w such that

$$l(w'_2, W(G)) = l(w'_1, W(G)) + l(w_0(P), W(G))$$

then $w_1 = w'_1$ and $w_2 = w'_2$.

Let L_r denote the (very) ample generator of Pic $H/P_r \approx \mathbf{Z}$. Then $H^0(H/P_r, L_r)$ acquires a canonical (left) H-module structure and, in fact, it is the dual of $\overset{r}{\Lambda} V$. Let us denote by

$$p_{(j)} = p_{(j_1, \ldots, j_r)}, (j) = (j_1, \ldots, j_r) \in I_m(r)$$

the canonical dual basis in $H^0(H/P_r, L_r)$, so that we have

$$\langle e_{(i)}, p_{(j)} \rangle = \delta_{(i), (j)} \text{ (Kronecker delta)}, e_i = e_{i_1} \Lambda \ldots \Lambda e_{i_r}.$$

The $p_{(j)}$ are known as the *Plücker coordinates*. We observe the following:

(i) $p_{(m-r+1, \ldots, m)}$ is a *highest weight vector in* $H^0(H/P_r, L_r)$ and its weight is $i(\omega_r)$, where i denotes the Weyl involution, i.e. $i = -w_0$, w_0 being the element of largest length in $W(H)$.

(ii) $p_{(1, \ldots r)}$ is a *lowest weight vector in* $H^0(H/P_r, L_r)$ and its weight is $-\omega_r$.

Now we have two natural ways of indexing in a more intrinsic manner the weight vectors in $H^0(H/P_r, L_r)$. For this we observe that the subgroup of $W(H)$ which fixes the 1-dimensional subspace spanned by $p_{(m-r+1, \ldots, m)}$ is $W(i(P_r))$, $i(P_r)$ being the maximal parabolic subgroup, canonically associated to $i(\omega_r)$, in the sense described above. We write this property as follows (by abuse of notation):

$$\tau \in W(H), \ \tau \cdot p_{(m-r+1, \ldots, m)} = p_{(m-r+1, \ldots, m)} \Longleftrightarrow \tau \in W(i(P_r)).$$

Similarly, one finds that

$$\tau \in W(H), \tau \cdot p_{(1, \ldots, r)} = p_{(1, \ldots, r)} \Longleftrightarrow \tau \in W(P_r).$$

Hence, we write

$$\{f_\tau\}, \tau \in W(H)/W(i(P_r)), \quad f_\tau = \tau \cdot p_{(m-r+1, \ldots, m)}$$

$$\{p_\tau\}, \tau \in W(H)/W(P_r); \quad p_\tau = \tau \cdot p_{(1, \ldots, r)}$$

$$\{e_\tau\}, \ \tau \in W(H)/W(P_r), \quad e_\tau = \tau \cdot e_{(1, \ldots, r)} = \tau \cdot (e_1 \Lambda \ldots \Lambda e_r)$$

(the elements f_τ, p_τ, e_τ are well-defined up to scalar multiples). The $\{p_\tau\}$, $\{f_\tau\}$ are two natural ways of indexing the weight vectors in $H^0(H/P_r, L_r)$. We see that

$$\langle e_\tau, p_{\tau'} \rangle = \delta_{\tau, \tau'} \text{ (Kronecker delta)}.$$

One has now the following properties, which hold in a more general context (Seshadri [22] where the indexing similar to $\{f_\tau\}$ is considered):

(i) For $\tau =$ identity (i.e. the class $W(i(P_r))$, f_τ is a *highest weight vector.*

(ii) $f_{\tau_1} \mid X(\tau_2 w_0,\ H/P_r)$ is not identically zero $\Longleftrightarrow \tau_1 \geqslant \tau_2$ in $W(H)/W(i(P_r))$ (one observes that the Schubert varieties in H/P_r can be written as $X(\tau w_0,\ H/P_r)$, $\tau \in (W(H)/W(i(P_r))$, but then

$$\tau_1 \geqslant \tau_2 \text{ in } W(H)/W(i(P_r)) \Longleftrightarrow X(\tau_1 w_0,\ H/P_r) \subseteq X(\tau_2 w_0,\ H/P_r)$$

(a) For $\tau =$ identity, p_τ is a *lowest weight vector.*

(b) $\tau_1 \leqslant \tau_2$ in $W(H)/W(P_r) \Longleftrightarrow p_{\tau_1} \mid X(\tau_2, H/P_r) \neq 0$.

Now (b) takes the explicit form as follows: Let $\tau \in W(H)/W(P_r) \approx I_m(r)$ and $\tau = (i) = (i_1, \ldots, i_r) \in I_m(r)$. Then if $p_{(j)}$, $(j) \in I_m(r)$ denotes a Plücker coordinate as above, we have

$$p_{(j)} \mid X(\tau,\ H/P_r) \neq 0 \Longleftrightarrow (j) \leqslant (i).$$

We see that the set

$$\{x \in H/P_r \mid p_{(m-r+1,\ldots,m)}(x) \neq 0\}$$

is the *big cell* in H/P_r. We call the set

$$\{x \in H/P_r \mid p_{(1,\ldots,r)}(x) \neq 0\}$$

the opposite big cell in H/P_r.

2. Determinantal varieties and Grassmannians (Hochster [9]; Musili [19])

We keep the basic notations of section 1. Let $M_{m,r}$ denote the set of $(m \times r)$ matrices with entries in K, $m > r$. Let V be as in section 1 i.e. an m-dimensional vector space over K with a standard basis $e_1, \ldots, e_m$, such that its elements are denoted by column vectors etc. Let $Z = (Z_1, \ldots, Z_r) \in M_{m,r}$, Z_i being the ith column of Z. Then we can identify Z_i with an element of V. Let φ_1 be the mapping

$$\varphi_1 : M_{m,r} \to \overset{r}{\Lambda} V,\ Z \to Z_1 \Lambda \ldots \Lambda Z_r.$$

Let $p_{(i)}$, $(i) = (i_1, \ldots, i_r) \in I_m(r)$, be as in section 1 the canonical dual basis in the dual of $\overset{r}{\Lambda} V$ (Plücker coordinates). We see that

$$q_{(i)} = (p_{(i)} \circ \varphi_1)(Z) = \begin{cases} \text{determinant of the } (r \times r) \text{ minor of } Z \text{ formed by} \\ \text{the rows corresponding to the indices } i_1, \ldots, i_r. \end{cases}$$

Multiplication by an $(m \times m)$ matrix (resp. $(r \times r)$ matrix) on the left (resp. on the right) gives a left (resp. right) action of $GL(m)$ (resp. $GL(r)$) on $M_{m,r}$. We see that φ_1 is $GL(m)$ equivariant. Let $M^s_{m,r}$ denote the subset of $M_{m,r}$ formed by matrices of rank r. Then φ_1 induces a surjective morphism

$$\varphi_2 : M^s_{m,r} \to H/P_r \ (H/P_r \text{ as in section 1})$$

and in fact the Grassmannian H/P_r is the orbit space $M^s_{m,r}/GL(r)$. We see that the inverse image by φ_2 of the *opposite big cell* in H/P_r is the following set

$$\left\{ Z \in M^s_{m,r} \,\middle|\, q_{(1,\ldots,r)}(Z) \neq 0 \right\}.$$

From this it is immediate that the restriction of φ_2 to the following subset of $M^s_{m,r}$ i.e.

$$\begin{pmatrix} Id_r \\ Y \end{pmatrix}, \quad \begin{cases} Id_r = r \times r \text{ identity matrix} \\ Y \in M_{m-r,r} \end{cases}$$

is an isomorphism onto the opposite big cell in H/P_r.

Let $m = n+r$ and $n \geqslant r$. Let ν be the closed immersion

$$\nu : M_{n,r} \to M_{n+r,r}$$

defined by

$$X \in M_{n,r}, \ X \to \begin{pmatrix} Id_r \\ JX \end{pmatrix}, \ J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} (n \times n \text{ matrix}).$$

We note that $\nu(M_{n,r}) \subset M^s_{n+r,r}$ and that $\varphi_2 \circ \nu$ is an isomorphism of $M_{n,r}$ on to the opposite big cell in H/P_r.

Definition 2.1. (i) Let $(i) = (i_1, \ldots, i_s)$ and $(j) = (j_1, \ldots, j_t)$ be two sets of integers. We write $(i) \leqslant (j)$ if $s \geqslant t$ and $i_k \leqslant j_k$, $k \leqslant t$, i.e.

$$\begin{matrix} i_1 \leqslant j_1 \\ \vdots \quad \vdots \\ i_t \leqslant j_t \\ \vdots \\ i_s \end{matrix}$$

If we have pairs of sets of integers $((i), (j))$ and $((i)', (j'))$, $(i) = (i_1, \ldots, i_s)$, $(i)' = (i_1', \ldots, i_{s'}')$, $(j) = (j_1, \ldots, j_t)$, $(j)' = (j_1', \ldots, j_{t'}')$, we write $((i), (j)) \leqslant ((i)', (j)')$ if $(i) \leqslant (i)'$ and $(j) \leqslant (j)'$.

(ii) Let $(i)=(i_1, \ldots, i_k) \in I_n(k)$, $(j)=(j_1, \ldots, j_k) \in I_r(k)$ with $0 \leqslant k \leqslant r$ (if $k=0$, (i) and (j) are supposed to be empty). We denote by $p_{(i),(j)} : M_{n,r} \to K$ the function defined by $Z \to p_{(i),(j)}(Z)$ where

$$p_{(i),(j)}(Z) = \begin{cases} \text{determinant of the } (k \times k) \text{ minor of } Z \text{ formed by the rows} \\ \text{and columns corresponding to the indices in (i) and} \\ (j) \text{ respectively. If } k=0,\ p_{(i),(j)}=1. \end{cases}$$

Definition 2.2. Let $(\lambda)=(\lambda_1, \ldots, \lambda_r) \in W(H)/W(P_r) \approx I_m(r)$ where $I_m(r)=\{(i_1, \ldots i_r) \mid 1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant m\}$.

We call the pair $((\alpha), (\beta))$ (resp. $((i), (j))$), $(\alpha) \in I_n(k)$, $(\beta) \in I_r(k)$, $0 \leqslant k \leqslant r$ (resp. $(i) \in I_n(k')$, $(j) \in I_r(k'))$, $0 \leqslant k' \leqslant r$) defined below, as the *canonical pair* (resp. the *canonical dual pair*) *associated to* (λ):

Firstly the integers k and k' are defined by

$\lambda_k \leqslant n$, $\lambda_{k+1} > n$:

$\lambda_{r-k'} \leqslant r$, $\lambda_{r-k'+1} > r$ (recall that $m=n+r$). Now set

$(\alpha) =$ $(\lambda_1, \ldots, \lambda_k)$

$(\beta) =$ complement of $(m+1-\lambda_{k+1}), (m+1-\lambda_{k+2}), \ldots, (m+1-\lambda_r)$ in $(1, \ldots, r)$ arranged in the ascending order.

$(i) =$ $(m+1-\lambda_{r-k'+1}), (m+1-\lambda_{r-k'+2}), \ldots, (m+1-\lambda_r)$ arranged in the ascending order.

$(j) =$ complement of $(\lambda_1, \ldots, \lambda_{r-k'})$ in $(1, \ldots, r)$ arranged in the ascending order.

Note: (i) k (resp. k') is $=0$ means that (α) and (β) (resp. (i) and (j)) are empty.

(ii) when $r=n$, we have $k'=n-k$ and

$(\beta) =$ complement of $(i_1, \ldots, i_{n-k})$ in $(1, \ldots, n)$ arranged in the increasing order.

$(\alpha) =$ complement of $(j_1, \ldots j_{n-k})$ in $(1, \ldots, n)$ arranged in the increasing order.

Lemma 2.1. (i) Let $(\lambda) \in I_m(r)$. Then if $q_{(\lambda)} : M_{n+r,r} \to K$ is the function defined by: $Z \in M_{n+r}$, $q_{(\lambda)}(Z) =$ determinant of the $(r \times r)$ minor of Z formed by rows corresponding to the indices $\lambda_1, \ldots, \lambda_r$, we have

$$q_{(\lambda)} \circ \nu = p_{(i),(j)},$$

where $((i), (j))$ is the *canonical dual pair* associated to (λ).

(ii) Let $(\lambda), (\lambda)' \in I_m(r)$ and $((\alpha), (\beta))$ (resp. $((i), (j))$), $((\alpha)', (\beta)')$ (resp. $((i)', (j)')$) the *canonical pairs* (resp. canonical dual pairs) associated respectively to (λ), $(\lambda)'$. Then we have a bijective map φ (resp. φ') of $W(H)/W(P_n)$ on to the set of canonical pairs (resp. canonical dual pairs) given by

$$\varphi((\lambda)) = ((\alpha), (\beta)) \text{ (resp. } \varphi'((\lambda)) = ((i), (j)).$$

Further

$$((\alpha), (\beta)) \leqslant ((\alpha)', (\beta)') \Longleftrightarrow (\lambda) \leqslant (\lambda)'$$

$$\text{resp. } ((i), (j)) \leqslant ((i)', (j)') \Longleftrightarrow (\lambda)' \leqslant (\lambda)).$$

i.e., φ (resp. φ') is order preserving (resp. order reversing).

Proof. (i) Let $W \in M_{n+r, r}$ be the element defined by

$$W = \begin{pmatrix} Y \\ J_2 \end{pmatrix}, \quad Y \in M_{n, r}, \quad J_2 = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & 0 \end{pmatrix} \ (r \times r \text{ matrix}).$$

We claim that we have the following:

$$(*) \quad \begin{cases} \text{Let } (\lambda) = (\lambda_1, \ldots, \lambda_r) \in I_{n+r}(r); \text{ then } q_{(\lambda)}(W) = \pm p_{(\alpha), (\beta)}(Y), \text{ where} \\ ((\alpha), (\beta)) \text{ is the canonical pair associated to } (\lambda) \ (cf. \text{ Definition 2.2 above}). \end{cases}$$

The relation (*) states that the determinant of the minor of W formed by the rows corresponding to the indices of (λ) is equal to the determinant (upto sign) of the minor of Y formed by the rows and columns corresponding to the indices in (α) and (β) respectively. We leave the verification of (*) as an exercise.

Let us set $Y = J_1 Z$, $J_1 = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & 0 \end{pmatrix}$ $(n \times n$ matrix). We see that

$$\begin{pmatrix} I \\ Z \end{pmatrix} = \begin{pmatrix} 0 & J_2 \\ J_1 & 0 \end{pmatrix} \begin{pmatrix} Y \\ J_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & J_2 \\ J_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & 0 \end{pmatrix} \ (m \times m \text{ matrix})$$

so that the matrix $\begin{pmatrix} I \\ Z \end{pmatrix}$ is obtained from $\begin{pmatrix} Y \\ J_2 \end{pmatrix}$ by changing the lth row to the $(n+r+1-l)$th row, $1 \leqslant l \leqslant n+r$. Thus (*) and this observation prove (i).

The assertion (ii) is left as an easy exercise.

Definition 2.3. (i) Let $R_{n, r}$ denote the polynomial algebra over K formed by nr indeterminates so that $M_{n, r}$ is the set of K-valued points of Spec $R_{n, r}$ and we refer to $R_{n, r}$ as the coordinate ring of $M_{n, r}$. Let $\triangle_k$ denote the ideal (called a *determinanta*

ideal) in $R_{n,r}$ generated by $p_{(i),(j)}$ (*cf.* Def. 2.1 for the definition of $p_{(i),(j)}$) such that the number of indices in (i) (= that of (j)) is $> k$ or equivalently

$$((1, \ldots k), (1, \ldots, k)) \not\leq ((i), (j)).$$

We denote by D_k the zero set (in $M_{n,r}$) of the ideal $\triangle_k$; D_k is called a *determinantal variety* (we shall see that D_k is a variety). We see that D_k is defined by the vanishing of all $(d \times d)$ minors, $d \geqslant k+1$.

(ii) We call an element in $R_{n,r}$ of the form

$$p_{(i),(j)}\, p_{(i)',(j)'}\, p_{(i)'',(j)''} \ldots$$

a *standard monomial in* $R_{n,r}$ *if*

$$((i), (j)) \leqslant ((i)', (j)') \leqslant ((i)'', (j)'') \ldots$$

i.e.
$$\begin{cases} (i) \leqslant (i)' \leqslant (i)'' \ldots \\ (j) \leqslant (j)' \leqslant (j)'' \ldots \end{cases}$$

The number of $p_{(i),(j)}$ in this expression is called the *length* of this standard monomial.

Theorem 2.1. We keep the notations above as well as the basic notations of section 1.

Then we have the following:

(i) We have a canonical identification of $W(H)/W(P_r) \approx I_m(r)$ with the set of pairs of elements of the form $((\alpha), (\beta))$ (resp. $(i), (j)$)

$$(\alpha) \in I_n(k), (\beta) \in I_r(k), 0 \leqslant k \leqslant r$$
(resp. $(i) \in I_n(k')$, $(j) \in I_r(k')$, $0 \leqslant k' \leqslant r$) (recall $m = n + r$).

In fact if $\tau = (\lambda) = (\lambda_1, \ldots, \lambda_r) \in W(H)/W(P_r)$, then the pair $((\alpha), (\beta))$ (resp. $((i), (j))$) is the *canonical pair* (resp. *canonical dual pair*) associated to τ (*cf.* Def. 2.2). This identification preserves (resp. reverses) the partial orders.

(ii) For $\tau \in W(H)/W(P_r)$, let $D(\tau) = D((i), (j))$ denote the subvariety of $M_{n,r}$, obtained as the intersection of $X(\tau, H/P_r)$ with the opposite big cell in H/P_r, identified with $M_{n,r}$ as above, $((i), (j))$ being the canonical dual pair associated to τ. Then the ideal $I(D(\tau))$ of the subvariety $D(\tau)$ of $M_{n,r}$ is the ideal in $R_{n,r}$ generated by

$$\{p_{(i)',(j)'}\},\ ((i), (j)) \not\leq ((i)', (j)').$$

In particular, the determinantal ideal $\triangle_k = I(D((i), (j)))$, $(i) = (j) = (1, \ldots, k)$, is a prime ideal in $R_{n,r}$ and $D_k = D((1, \ldots, k), (1, \ldots, k))$ is a variety.

(iii) Let $R(\tau) = R((i), (j))$ be the coordinate ring of the subvariety $D(\tau) = D((i), (j))$ of $M_{n,r}$ (as in (ii) above) so that

$$R(\tau) = R_{n,r}/I(D(\tau)).$$

Then distinct standard monomials of the form

$$p_{(i)',(j)'}\, p_{(i)'',(j)''}\, p_{(i)''',(j)'''} \cdots,$$
$$((i),(j)) \leqslant ((i)',(j)') \leqslant ((i)'',(j)'') \leqslant \cdots$$

form a basis of (the underlying vector space of) $R(\tau)$. In particular $R(\tau)$, where the canonical dual pair associated to τ is $((1, \ldots k), (1, \ldots k))$, has a basis formed of distinct standard monomials

$$p_{(i)',(j)'}\, p_{(i)'',(j)''} \cdots$$

such that the number of indices in $(i)'$ (and hence in $(j)'$, $(i)''$, $(j)''$ ... etc.) is $\leqslant k$.

Proof. The assertion (i) is just a restatement of Lemma 2.1 above and the identification of $W(H)/W(P_n)$ mentioned at the end of section 1.

One knows (*cf.* section 1) that the weight vectors of $H^0(H/P_r, L_r)$ (L_r—the ample generator of Pic H/P_r) can be indexed by p_τ, $\tau \in W(H)/W(P_r)$, such that the restriction of p_τ to $X(\rho, W(H)/W(P_r))$, $\rho \in W(H)/W(P_r)$ is zero if and only if $\tau \not\leqslant \rho$ (*cf.* section 1). We call a *standard monomial of* (*length l*) in $\{p_\tau\}$ an expression of the following form

$$p_{\tau_1} p_{\tau_2} \cdots p_{\tau_l},\ \tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_l$$

Let A be the homogeneous coordinate ring of H/P_r with respect to its canonical imbedding in $\mathbf{P}(\overset{r}{\Lambda} V)$. We have

$$A = \sum_{l=0}^{\infty} H^0(H/P_r, L_r^l).$$

Then the basic theorem of the Hodge-Young theory on Grassmannians (Hodge and Pedoe [12]; Musili [18]; Seshadri [22]) states that distinct standard monomials form a basis of A (as a vector space). Recall that $f = p_{(1, \ldots, r)}$, is a lowest weight vector, so that the set

$$U = \{x \in H/P_r \mid f(x) \neq 0\}$$

is the opposite big cell in H/P_r. Let $A_{(f)}$ denote the 'homogeneous localisation' of A with respect to f, i.e. $A_{(f)}$ is the subring of A_f (localisation of A with respect to f) generated by the elements $\{p_\tau/f\}$, $\tau \in W(H)/W(P_r)$. Let us call a *standard monomial in* $A_{(f)}$ an element which is either 1 or an expression of the form

$$\frac{p_{\tau_1}}{f} \cdot \frac{p_{\tau_2}}{f} \cdots \frac{p_{\tau_l}}{f},\ \tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_l,\ p_{\tau_i} \neq f,\ 1 \leqslant i \leqslant l.$$

We claim that distinct standard monomials in $A_{(f)}$ form a basis of $A_{(f)}$. It suffices to prove that they are linearly independent. To prove this we observe that if θ is a standard monomial in A, then θf^k is also a standard monomial in A, since $f = p_{(1, \ldots, r)}$ and $(1, \ldots, r)$ identified with an element of $W(H)/W(P_r)$ is the least element in $W(H)/W(P_r)$. Hence, if we have a linear combination of distinct standard monomials in $A_{(f)}$, multiplying it by a suitable power of f, makes it a linear combination of distinct standard monomials in A and from this it is immediate that distinct standard monomials in $A_{(f)}$ are linearly independent.

We have a canonical identification of $M_{n,r}$ with the opposite big cell U in H/P_r and $U = \text{Spec } A_{(f)}$. Hence we have a canonical identification of $A_{(f)}$ with the coordinate ring $R_{n,r}$ of $M_{n,r}$. In this identification, we see that p_τ identifies with $\pm p_{(i),(j)} \in R_{n,r}$, where $((i), (j))$ is the canonical dual pair associated to τ (*cf.* Def. 2.2 and Lemma 2.1). Thus this identification, together with the fact that distinct standard monomials in $A_{(f)}$ form a basis, proves the assertion (iii) of the above theorem for the case $D(\tau) = M_{n,r}$. The Hodge-Young theory implies in fact (Hodge [11]; Hodge and Pedoe [12]; Musili [18]) that in the homogeneous coordinate ring of the Schubert variety $X(\tau, W(H)/W(P_r))$, $\tau \in W(H)/W(P_r)$, distinct standard monomials of the form

$$p_{\tau_1} \cdots p_{\tau_l},\ \tau \geqslant \tau_1$$

form a basis and that the homogeneous ideal of $X(\tau, W(H)/W(P_r))$ is generated by $\{p_\alpha\}$, $\alpha \not\leqslant \tau$. From these the remaining claims in (iii) as well as (ii) follow easily. This proves the theorem.

Remark 2.1. We have deduced the above theorem as a consequence of the Hodge-Young theory on Grassmannians (Hodge and Pedoe [12]; Musili [18]; Seshadri [22]). We see in fact that the above theorem is essentially equivalent to the Hodge-Young theory on Grassmannians i.e. conversely we can deduce the Hodge-Young theory on Grassmannians from the above theorem. For example, in the proof of the theorem, we should have noted that we have in fact:

Distinct Standard monomials in A form a basis $\Longleftrightarrow$

Distinct Standard monomials in $A_{(f)}$ form a basis.

The assertion (iii) of the above theorem for the case $D(\tau) = M_{n,r}$ is proved in Doubilet-Rota-Stein (Doubilet *et al* [7]) and the couples

$$((i)', (j)') \leqslant ((i)'', (j)'') \leqslant \ldots$$

which figure in the definition of a standard monomial in $R_{n,r}$ (*cf.* Definition 2.1) are called by them 'double standard tableaux' generalizing the usual Young's notion of a standard tableau.

3. Some basic facts about the symplectic and special orthogonal groups.

We keep the basic notations of sections 1 and 2. Hereafter we take $m = 2n$ so that

$$H = SL(2n).$$

Let E_1, E_2 denote the $(2n \times 2n)$ matrices

$$E_1 = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix},\ J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot} & \\ 1 & & 0 \end{pmatrix} (n \times n \text{ matrix}).$$

We set

$$G_1 = Sp(2n) = \{A \in SL(2n) \text{ (or } GL(2n))/ {}^tAE_1A = E_1\}$$

$$G_2 = SO(2n) = \{A \in SL(2n)/ {}^tAE_2A = E_2\}, \text{ char } K \neq 2.$$

We see that

$${}^tAE_1A = E_1 \iff A = E_1^{-1} ({}^tA)^{-1} E_1 \text{ (note } E_1^{-1} = -E_1)$$

$${}^tAE_2A = E_2 \iff A = E_2 ({}^tA)^{-1} E_2 \text{ (note } E_2 = E_2^{-1}).$$

Let σ_1, σ_2 denote the involutory automorphisms in $GL(2n)$ (as well as their restrictions to $SL(2n)$) defined by

$$\sigma_1(A) = E_1 ({}^tA)^{-1} E_1^{-1},\ \sigma_2(A) = E_2 ({}^tA)^{-1} E_2.$$

We see that

$$G_1 = SL(2n)^{\sigma_1} = GL(2n)^{\sigma_1}$$

$$G_2 = SL(2n)^{\sigma_2} = GL(2n)^{\sigma_2} \cap SL(2n)$$

i.e., G_i is the fixed point set in $SL(2n)$, under σ_i, $i = 1, 2$. It is easily checked that the groups T_{2n}, $T(H)$ and $B(H)$ are stable under σ_i, $i = 1, 2$. We set

$$T(H)^{\sigma_i} = T(G_i),\ B(H)^{\sigma_i} = B(G_i),\ i = 1, 2.$$

One knows that $T(G_i)$ is a maximal torus in G_i and that $B(G_i)$ is a Borel subgroup in G_i, $i = 1, 2$. We see that $T(G_i)$ has the following form:

$$T_{2n}^{\sigma_1} = T_{2n}^{\sigma_2} = T(G_1) = T(G_2) = \begin{pmatrix} t_1 & & & 0 \\ & \cdot^{\cdot} & & \\ & & t_n^{-1} & \\ & & & \cdot^{\cdot} \\ 0 & & & t_1^{-1} \end{pmatrix}$$

Let Lie G_i denote the Lie algebra of $G_i (i=1, 2)$. We see that

$$\text{Lie } G_1 = \{A \in M_{2n} \mid E_1^{-1}\, {}^tAE_1 + A = 0\}.$$

$$\text{Lie } G_2 = \{A \in M_{2n} \mid E_2\, {}^tAE_2 + A = 0\}.$$

The adjoint representation of G_i on $L(G_i)$ is given by

$$g \in G_i,\ A \in \text{Lie } G_i, \text{ then } g\ o\ A = g\, A\, g^{-1}.$$

We set, for $i = 1, 2$

(i) $N(T(G_i)) =$ normaliser of $T(G_i)$ in G_i

(ii) $W(G_i) = N(T(G_i))/T(G_i)$.

(iii) (a) $X_0(T(G_i)) = \text{Hom}\,(T(G_i), G_m)$.

(b) $X(T(G_i)) = X_0(T(G_i)) \otimes_{\mathbf{Z}} \mathbf{R}$.

We note that the following hold:

I. $N(T(G_i)) \subset N(T(H)),\ i = 1, 2.$

This is a consequence of the fact that if $D \in GL(2n)$ is a diagonal matrix with *distinct* diagonal elements, then for $A \in SL(2n)$

$$ADA^{-1} \text{ is diagonal} \Longrightarrow A \in N(T(H)).$$

II. $N(T(H))$ is stable under σ_i and

$$N(T(G_i)) = N(T(H))^{\sigma_i},\ i = 1, 2$$

III. The canonical map

$$N(T(G_i))/T(G_i) \to N(T(H))/T(H)\ (i = 1, 2)$$

is an inclusion i.e. the Weyl group $W(G_i)$ of G_i $(i = 1, 2)$ can be identified canonically as a subgroup of the Weyl group of H.

IV. The involution σ_i induces an involution on $W(H)$ since σ_i leaves $N(T(H))$ stable as well as the subgroup $T(H)$ and this induced involution on $W(H)$ is the same for $i = 1, 2$. Let us denote by σ this induced involution on $W(H)$. It is checked easily that if $w = (a_1, \ldots, a_{2n}) \in W(H)$, then

$$\sigma(w) = (c_1, \ldots, c_{2n}),\ c_i = 2n+1-a_{2n+1-i}.$$

Let us denote by $\tilde{\sigma}$ the permutation i.e. the element of S_{2n}, defined by

$$\tilde{\sigma}(i) = 2n+1-i,\ 1 \leqslant i \leqslant 2n.$$

Then we have

$$\sigma(w) = \tilde{\sigma}\, w\, \tilde{\sigma}\ (\tilde{\sigma} = \tilde{\sigma}^{-1}), \text{ so that}$$

$$w \in (W(H))^{\sigma} \Longleftrightarrow w = \tilde{\sigma}\, w\, \tilde{\sigma}$$

$$\Longleftrightarrow a_i = 2n+1-a_{2n+1-i}$$

One sees that

(a) $W(G_1) = (W(H))^{\sigma}$ i.e. if $w = (a_1, \ldots, a_{2n}) \in W(H)$, then $w \in W(G_1) \Longleftrightarrow a_i = 2n+1-a_{2n+1-i}$.

(b) $W(G_2) = \{w \in (W(H))^{\sigma} = W(G_1)/w$ is an *even* permutation in $W(H)\}$.

V. The canonical involution on $X(T_{2n})$ (resp. $X_0(T_{2n})$, $X_0(H)$, $X(H)$) is independent of $i = 1, 2$ and we denote this by σ. With respect to the canonical basis $\{\epsilon_i\}$, $1 \leqslant i \leqslant 2n$ of $X(T_{2n})$ (section 1), σ takes the form

$$\sigma \epsilon_i = -\epsilon_{2n+1-i},\quad 1 \leqslant i \leqslant 2n.$$

Recall that we have identified $X(T(H))$ as the subspace of $X(T_{2n})$, spanned by $\epsilon_i - \epsilon$ (section 1). We have a canonical surjective linear map

$$\varphi : X(T_{2n}) \to X(T(G_1)) = X(T(G_2)).$$

We see that

$$\varphi(\epsilon_i) = -\varphi(\epsilon_{2n+1-i}),\ 1 \leqslant i \leqslant 2n.$$

We have then the following properties:

(a) σ leaves $R(H)$ (resp. $R^+(H)$) stable (identifying $R(H)$ with $\epsilon_i - \epsilon_j$, $i \neq j$)

(b) $\alpha, \beta \in R(H)$, then

$$\varphi(\alpha) = \varphi(\beta) \Longleftrightarrow \alpha = \sigma(\beta)$$

(c) φ is equivariant for the canonical actions of $W(G_1)$ on $X(T_{2n})$ and $X(T(G_1))$.

(d) the elements of $R(H)$ which are *fixed by* σ are

$$\pm(\epsilon_i - \epsilon_{2n+1-i}),\ 1 \leqslant i \leqslant n.$$

From these considerations, as well as the explicit nature of the adjoint representation of G_i on Lie G_i, $i = 1, 2$, we deduce that

(e) $R(G_1)$ (resp. $R^+(G_1)$) $= \varphi(R(H))$ (resp. $\varphi\ (R^+(H))$)

$R(G_2)$ (resp. $R^+(G_2)$) $= \varphi\ (R(H) - R(H)^\sigma)$ (resp. $\varphi\ R^+(H) - (R^+\ (H))^\sigma)$)

where $R(G_i)$ (resp. $R^+(G_i)$) denotes the set of roots (resp. positive roots) of G_i with respect to $T(G_i)$ and $B(G_i)$, $i = 1, 2$. We can state this relation as saying that $R(G_1)$ (resp. $R^+(G_1)$) can be identified with the orbit space $R(H)$ (resp. $R^+(H)$) modulo the action of σ and that $R(G_2)$ (resp. $R^+(G_2)$) can be identified with the orbit space under σ of $R(H)$ (resp. $R^+\ (H)$) minus the fixed point set under σ. We see now that $R^+(G_i)$ can be identified through φ with the following subsets of $X(T_{2n})$:

$$R^+(G_1) = \begin{cases} \epsilon_i - \epsilon_j, & 1 \leqslant i < j \leqslant n \\ \epsilon_i + \epsilon_j, & 1 \leqslant i < j \leqslant n \\ 2\epsilon_i & 1 \leqslant i \leqslant n \end{cases}$$

$$R^+(G_2) = \begin{cases} \epsilon_i - \epsilon_j, & 1 \leqslant i < j \leqslant n \\ \epsilon_i + \epsilon_j, & 1 \leqslant i < j \leqslant n \end{cases}$$

The canonical action of $W(G_1)$ on $R(G_1)$ can then be written explicitly as follows:

$$w = (a_1, \ldots, a_{2n}), \quad \begin{cases} w(\epsilon_i - \epsilon_j) = \eta_{a_i} - \eta_{a_j} \\ w(\epsilon_i + \epsilon_j) = \eta_{a_i} + \eta_{a_j} \\ w(2\epsilon_i) \quad\ = 2\eta_{a_i} \end{cases}$$

where

$$\eta_i = \epsilon_i,\ 1 \leqslant i \leqslant n;\ \eta_i = -\epsilon_{2n+1-i},\ n+1 \leqslant i \leqslant 2n.$$

The explicit description of the canonical action of $W(G_2)$ on $R(G_2)$ follows from above, since $W(G_2) \subset W(G_1)$ and $R(G_2) \subset R(G_1)$.

VI. Recall (section 1) that we have denoted by $\{s_i\}$, $1 \leqslant i \leqslant 2n-1$, the simple reflections in H, i.e. $\{s_i\}$ are the reflections with respect to the simple roots $\epsilon_i - \epsilon_{i+1}$, $1 \leqslant i \leqslant 2n-1$, so that s_i is the transposition $(i, i+1)$. One sees easily that the simple roots in $R(G_1)$ and $R(G_2)$ (with the identification as subsets of $X(T_{2n})$ as above) are given as follows:

$$\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3,\ \ldots,\ \epsilon_{n-1} - \epsilon_n,\ 2\epsilon_n \text{ in } R(G_1)$$
$$\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3,\ \ldots,\ \epsilon_{n-1} - \epsilon_n,\ \epsilon_{n-1} + \epsilon_n \text{ in } R(G_2).$$

Let us denote by $\{\theta_i\}_{1 \leqslant i \leqslant n}$ the simple reflections in $W(G_1)$ as well as $W(G_2)$; to be more precise, set:

$$\left.\begin{array}{l} \theta_i = \text{reflection with respect to } \epsilon_i - \epsilon_{i+1},\ 1 \leqslant i \leqslant n-1 \\ = \quad\ ,, \qquad\qquad ,, \qquad 2\epsilon_n,\ i = n \end{array}\right\} \text{ in } W(G_1)$$

$$\left.\begin{array}{l}\theta_i = \text{reflection with respect to } \epsilon_i - \epsilon_{i+1}, \quad 1 \leqslant i \leqslant n-1 \\ = \quad \text{,,} \quad \text{,,} \quad \epsilon_{n-1} + \epsilon_n, \quad i = n\end{array}\right\} \text{ in } W(G_2).$$

It is an easy exercise to deduce the following (Bourbaki [2], p. 53).

(a) $\left.\begin{array}{l}\theta_i = s_i\, \sigma(s_i) \\ = s_i\, s_{2n-i}\end{array}\right\}$, $1 \leqslant i \leqslant n-1$ in $W(G_1)$ and $W(G_2)$

(b) $\theta_n = s_n$ in $W(G_1)$ $(s_n = \sigma(s_n))$
$= \left.\begin{array}{l}\text{product of the transpositions: } (n-1, n+1)\ (n, n+2) \text{ or} \\ \text{equal to } s_{n-1}\, s_n\, s_{n-1}\, s_n\, s_{n+1}\, s_n\end{array}\right\}$ in $W(G_2)$.

Definition 3.1. (i) Let $w \in W(G_1)$ (resp. $W(G_2)$). Then we denote by $l(w, W(G_1))$ (resp. $l(w, W(G_2))$, the *length* of w *in* $W(G_1)$ (*resp.* $W(G_2)$) i.e. the length of a reduced decomposition of w with respect to the simple reflections $\{\theta_i\}$ in $W(G_1)$ (resp. $W(G_2)$). Since $W(G_i) \subset W(H)$, $i = 1, 2$, we denote by $l(w, W(H))$ the length of w in $W(H)$.

(ii) For $w = (a_1, \ldots, a_{2n}) \in W(H)$, we set

$$m(w) = \#\,\{i \mid i \leqslant n,\ w(i) = a_i > n\}$$

Lemma 3.1. An element $w \in W(G_1)$ is in $W(G_2)$ if and only if $m(w)$ is *even* i.e.

$$W(G_2) = \{w \mid w \in W(H),\ w \text{ is } \sigma\text{-invariant and } m(w) \text{ is even}\}.$$

Proof. We leave this as an exercise.

Proposition 3.1. We have the following

(i) $l(w, W(H)) = 2l(w, W(G_1)) - m(w)$, $w \in W(G_1)$

(ii) $l(w, W(H)) = 2l(w, W(G_2)) + m(w)$, $w \in W(G_2)$.

Proof. This is an easy consequence of the fact given in V(e) above, namely that $R(G_1)$ (resp. $R(G_2)$) can be identified with the orbit space of $R(H)$ (resp. $R(H)$—$R(H)^\sigma$) under the action of σ etc. For w as in the proposition, we set

$$S(w, G_i) = \{\beta \in R^+(G_i) \mid w(\beta) < 0\},\ i = 1 \text{ or } 2.$$

$$S(w, H) = \{\beta \in R^+(H) \mid w(\beta) < 0\}.$$

We have

$$l(w, W(H)) = \#\, S(w, H),\ l(w, W(G_i)) = \#\, S(w, G_i),\ i = 1, 2.$$

Consider the canonical map

$$\varphi : X(T_{2n}) \to X(T(G_1)) = X(T(G_2)) \text{ (cf. V above).}$$

Then φ induces *surjective* maps

$$\varphi : R(H) \to R(G_1) \text{ (resp. } \varphi : R(H) - R(H)^\sigma \to R(G_2)).$$

The mapping φ has the property:

$$\alpha \in R(H), \text{ then } \alpha > 0 \iff \varphi(\alpha) > 0.$$

From this we see that φ induces a surjective map

$$\varphi : S(w, H) \to S(w, G_1)$$

$$(\text{resp. } \varphi : S(w, H) - S(w, H)^\sigma \to S(w, G_2)).$$

We see easily that σ leaves $S(w, H)$ stable. Besides,

$$(S(w, H))^\sigma = \{\alpha / \alpha = \epsilon_i - \epsilon_{2n+1-i},\ 1 \leqslant i \leqslant n \text{ and } w(\alpha) < 0 \text{ in } R(H)\}.$$

Now if $\alpha = \epsilon_i - \epsilon_{2n+1-i}$, then $w(\alpha) = \epsilon_{a_i} - \epsilon_{a_{2n+1-i}}$. Hence

$$w(\alpha) < 0 \text{ in } R(H) \iff a_{2n+1-i} < a_i,\ i \leqslant n.$$

But $a_{2n+1-i} = 2n+1-a_i$ since $w \in W(G_1)$ (*cf.* IV above).

Thus

$$w(\alpha) < 0 \iff 2n+1 < 2a_i,\ 1 \leqslant i \leqslant n$$

$$\text{i.e. } n < a_i, \quad 1 \leqslant i \leqslant n$$

so that

$$\# (S(w, H))^\sigma = m(w) \text{ (*cf.* (ii), Definition 3.1).}$$

We see that

$$\# S(w, H) = 2(\# S(w, G_1)) - \# (S(w, H))^\sigma.$$

$$(\text{resp. } 2(\# S(w, G_2)) + \# (S(w, H))^\sigma).$$

The proposition is now an immediate consequence.

For $w \in W(H)$, we denote by $C(w, H/B(H))$ the *Schubert cell* in $H/B(H)$ defined by w i.e., the subset $B(H)w\, e_{B(H)}$ in $H/B(H)$. If $w \in W(G_1)$ (resp. $W(G_2)$), we denote by $C(w, G_1/B(G_1))$ (resp. $C(w, G_2/B(G_2))$) the Schubert cell in $G_1/B(G_1)$ (resp. $G_2/B(G_2)$), *Lemma* 3.2. For $w \in W(G_i)$ $(i = 1, 2)$, the Schubert cell $C(w, H/B(H))$ is stable under σ_i and we have

$$C(w, H/B(H))^{\sigma_i} = C(w, G_i/B(G_i)).$$

Proof. The fact that $C(w, H/B(H))$ is stable under σ_i $(i = 1, 2)$ is immediate. Let $B(H)^u$ denote the unipotent part of $B(H)$. Let B_1 denote the isotropy subgroup of $B(H)^u$ at $w \in C(w, H/B(H))$. Then since $\sigma_i(w) = w$, we see that σ_i leaves B_1 stable. We have

$$(*) \qquad B(H)^u = \prod_{\alpha \in R^+(H)} U_\alpha,\ U_\alpha \approx \mathbf{G}_a$$

U_α being the 1-dimensional subgroup of $B(H)^u$ canonically associated to $\alpha \in R^+(H)$. We see that

$$B_1 = \prod_{\substack{\alpha \in R^+(H), \\ w^{-1}(\alpha) > 0}} U_\alpha$$

Let

$$B_2 = \prod_{\substack{\alpha \in R^+(H), \\ w^{-1}(\alpha) < 0}} U_\alpha$$

By the 'uniqueness' of the decomposition (*) (Chevalley [4]), we see that σ_i leaves B_2 stable. Now if $x \in C(w, H/B(H))$, it has a representation $x = bw$, $b \in B_2$, b unique. Hence $\sigma_i(x) = x$ is equivalent to $\sigma_i(b) = b$ and the lemma follows.

4. The varieties G/Q, G of type $\mathbf{C}_n$ or $\mathbf{D}_n$ and Q the maximal parabolic subgroup associated to a right end root

As in section 1, let P_n denote the maximal parabolic subgroup in H, $H = SL(2n)$ associated to the simple root $\alpha_n = \epsilon_n - \epsilon_{n+1}$. Then P_n is of the form

$$\left(\begin{array}{c|c} (n\times n) & * \\ \hline 0 & (n\times n) \end{array}\right).$$

We see that the involutions σ_1, σ_2 leave P_n stable (the other maximal parabolic subgroups of H are *not* left stable by σ_i, in fact $\sigma_1(P_r)$ (resp. $\sigma_2(P_r)$) $= i(P_r)$, i as in section 1), so that σ_i induces canonical involutions (denoted by the same σ_i) on H/P_n, $i = 1, 2$.

Let Z be the subgroup of H of the form

$$Z = \left\{ \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} \Big/ \begin{array}{l} Id_n = (n\times n) \text{ identity matrix} \\ Y \in M_n \text{ (space of } n \times n \text{ matrices)} \end{array} \right\}$$

The canonical morphism $H \to H/P_n$ induces a canonical morphism

$$\psi : Z \to H/P_n.$$

It is now easy to check the following:

(i) ψ is an open immersion and $\psi(Z)$ identifies with the *opposite big cell* in H/P_n. We see also that if we identify Z canonically with the following subset of $M_{2n,\,n}$.

$$Z \approx \left\{ \begin{pmatrix} Id_n \\ Y \end{pmatrix} \Big/ \; Y \in M_n \right\},$$

then $Z \subset M^s_{2n,\,n}$ (notation of section 2) and the morphism $Z \to H/P_n$ induced by the canonical morphism $\varphi_2 : M^s_{2n,\,n} \to H/P_n$ coincides with the morphism ψ above.

(ii)
$$\sigma_1 \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} = \begin{pmatrix} Id_n & 0 \\ J\,{}^tYJ & Id_n \end{pmatrix}$$

$$\sigma_2 \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} = \begin{pmatrix} Id_n & 0 \\ -J\,{}^tYJ & Id_n \end{pmatrix}$$

In particular, the involution $\sigma_i (i = 1, 2)$ leaves stable Z and the opposite big cell $\psi(Z)$ in H/P_n. Further

$$Z^{\sigma_1} = \left\{ \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} \Big| \begin{array}{l} Y \in M_n \\ J\,{}^tYJ = Y \end{array} \right\},$$

$$Z^{\sigma_2} = \left\{ \begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} \Big| \begin{array}{l} Y \in M_n \\ -J\,{}^tYJ = Y \end{array} \right\}.$$

If we set $Y = JX$, then we have

(a) $J\,{}^tYJ = Y \iff {}^tX = X$

(b) $-J\,{}^tYJ = Y \iff {}^tX = -X.$

Thus if we identify Z or M_n with the opposite big cell in H/P_n by the mapping $M_n \to H/P_n$ defined by

$$X \in M_n,\; X \to \psi \begin{pmatrix} Id_n & 0 \\ JX & Id_n \end{pmatrix} \in H/P_n,$$

we see that the set of fixed points of the opposite big cell in H/P_n under σ_1 (resp. σ_2) can be identified with the set of symmetric (resp. skew-symmetric) matrices in M_n.

We see immediately that

$$Y = J\,{}^tYJ \iff (Id_n,\, {}^tY) \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \begin{pmatrix} Id_n \\ Y \end{pmatrix} = 0$$

$$Y = -J\,{}^tYJ \iff (Id_n,\, {}^tY) \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \begin{pmatrix} Id_n \\ Y \end{pmatrix} = 0.$$

Thus the condition that $Y = J\,{}^tYJ$ (resp. $Y = -J\,{}^tYJ$) means that the n-dimensional linear subspace of the $2n$-dimensional linear space V, represented by the following point of H/P_n, namely

$$\psi\begin{pmatrix} Id_n & 0 \\ Y & Id_n \end{pmatrix} = \begin{cases} \text{the } n\text{-dimensional linear subspace spanned by the} \\ \text{columns of } \begin{pmatrix} Id_n \\ Y \end{pmatrix} \in M_{2n,\,n} \text{ the columns representing points} \\ \text{of } V \text{ endowed with its standard basis } e_1, \ldots, e_{2n} \text{ (section 2)} \end{cases}$$

is a maximal *totally isotropic subspace* for the skew-symmetric (resp. symmetric) form on V represented by E_1 (resp. E_2). One knows that the set of points in H/P_n, represented by maximal totally isotropic subspaces with respect to E_1, is a closed subvariety isomorphic to G_1/Q, where Q is the maximal parabolic subgroup corresponding to the ' right end root ' in the Dynkin diagram of G_1 (i.e. the root $2\epsilon_n$). Further we note that

$$\dim G_1/Q = \frac{n(n+1)}{2} \left(= \text{dimension of the set of } (n \times n) \text{ symmetric matrices}\right)$$

This dimension can be calculated, for example, by noting that

$$\dim\ G_1/Q = \#\ R^+(G_1) - \#\ R^+(Q_0)\ R^+(Q_0) \text{ denotes the set of positive roots of the semi-simple part } Q_0 \text{ of } Q\text{; note that } Q_0 \text{ is of type } \mathbf{A}_{n-1}).$$

Similarly, one knows that the set of points in H/P_n, represented by maximal totally isotropic subspaces with respect to E_2, is a union of two closed subvarieties, each of which is isomorphic to G_2/Q, Q being a maximal parabolic subgroup in Q corresponding to one of the right end roots in the Dynkin diagram of G_2 (i.e. the roots $\epsilon_{n-1} - \epsilon_n$ and $\epsilon_{n-1} + \epsilon_n$). Further

$$\dim G_2/Q = \frac{n(n-1)}{2} = \text{dimension of the set of } (n \times n) \text{ skew-symmetric matrices.}$$

From these facts, we conclude easily the following:

Let Sym M_n (resp. $Sk\ M_n$) denote the set of symmetric (resp. skew-symmetric) in the set M_n of $(n \times n)$ matrices. Let Δ_0 and Δ denote the morphisms

$$\Delta_0 : M_n \to H/P_n$$

$$\Delta : \text{Sym } M_n \to H/P_n \text{ (resp. } Sk\ M_n \to H/P_n)$$

defined by

$$X \in M_n,\ X \longmapsto \psi\begin{pmatrix} Id_n & 0 \\ J & Id_n \end{pmatrix},$$

$$X \in \text{Sym } M_n \text{ (resp. } Sk\ M_n),\ X \longmapsto \begin{pmatrix} Id_n & 0 \\ J & Id_n \end{pmatrix}.$$

Then the image of Δ is an *open* subvariety of the closed subvariety G_1/Q (resp. G_2/Q) of H/P_n, Q being the maximal parabolic in G_1 corresponding to the right end root (resp. to a right end root of G_2).

Recall that for $(\lambda) = (\lambda_1, \ldots, \lambda_n) \in I_{2n}(n)$, we had denoted by $p_{(\lambda)}$ the 'Plücker coordinate' associated to (λ) and it is an element of $H^\circ(H/P_n, L_n)$, L_n being the ample generator of Pic H/P_n. If we take the canonical order preserving isomorphism $I_{2n}(n) \approx W(H)/W(P_n)$, we have seen (section 1) that if $\tau = (\lambda) \in W(H)/W(P_n)$, then

$$p_\tau = p_{(\lambda)} = \tau \cdot p_{(1, \ldots, n)}.$$

Let ((i), (j)) be the *canonical dual pair* associated to (λ) (*cf.* Def. 2.2). Then we have seen (*cf.* Lemma 2.1 and proof of Th. 2.1) that the restriction of $p_{(\lambda)}$ to M_n (identified as the opposite big cell of H/P_n through Δ_0) or to be more precise $p_{(\lambda)}/f(f = p_{(1, \ldots, n)})$, can be identified with the function $p_{(i), (j)}$ on M_n (*cf.* Def. 2.1).

Then we set the following:

(i) L'_n = restriction of the line bundle L_n on H/P_n to G_1/Q (resp. G_2/Q)

(ii) $p'_{(\lambda)}$ for $(\lambda) \in I_{2n}(n)$, is the restriction of $p_{(\lambda)}$ to G_1/Q (resp. G_2/Q), so that $p'_{(\lambda)}$ is a section of the line bundle L'_n.

(iii) $p'_{(i), (j)}$ is the restriction of $p_{(i), (j)}$ to Sym M_n (resp. Sk M_n).

(iv) f = a lowest weight vector in $H^\circ(H/P_n, L_n)$ and f = the restriction of f' to G_1/Q (resp. G_2/Q).

Then we have the following:

(i) The section $p'_{(\lambda)} \in H^\circ(G_1/Q, L'_n)$, $(\lambda) \in I_{2n}(n)$ is non-zero. This happens because the restriction of $p'_{(\lambda)}$ to Sym M_n (identified as an open subset of G_1/Q through $\triangle$) is $p'_{(i), (j)}$, ((i), (j)) being the canonical dual pair associated to (λ) and $p'_{(i), (j)} \neq 0$. Since f is a *lowest weight* vector in $H^\circ(H/P_n, L_n)$, the one dimensional space spanned by f is $B(H)^-$ stable, where $B(H)^-$ is the Borel subgroup of H *opposite to* $B(H)$ (consisting of the lower triangular matrices in H). One sees that $B(H)^-$ is stable under σ_1 and that $(B(H)^-)^{\sigma_1} = B(H)^- \cap G_1$ is a Borel subgroup in G_1 and one concludes easily that $B(H)^- \cap G_1$ is in fact the Borel subgroup $B(G_1)^-$, opposite to $B(G_1)$. We see then that f' is a lowest weight vector in $H^\circ(G_1/Q, L'_n)$. The weight of f is $-(\epsilon_1 + \ldots + \epsilon_n)$. Hence the weight of f' is also $-(\epsilon_1 + \ldots + \epsilon_n)$ (identifying $X(T(G_1))$ canonically as a subspace of $X(T_{2n})$). We see that

$$-(\epsilon_1 + \ldots + \epsilon_n) = (w_0)_{G_1}(\omega_n)$$

where ω_n is the fundamental weight associated to the right end root i.e. $\langle \omega_n, 2\epsilon_n{}^v \rangle = 1$ and $(w_0)_{G_1}$ is the element of $W(G_1)$ of largest length [Bourbaki [2]; note that $(w_0)_{G_1} = -1$ on $X(T(G_1))$]. Thus we conclude that the highest weight vector in

$H^\circ(G_1/Q, L'_n)$ has weight ω_n. We see then that L'_n is the ample generator of Pic G_1/Q so that $H^\circ(G_1/Q, L'_n)$ is the fundamental representation with highest weight $\omega_n = i(\omega_n)$. Since Sym M_n is precisely the set of points where f' does not vanish, we see that Sym M_n can be identified (through $\triangle$) with the opposite big cell in G_1/Q,

(ii) Let $p'_{(\lambda)} \in H^\circ(G_2/L_n')$ and $p'_{(i),(j)}$ the restriction of $p'_{(\lambda)}$ to Sk M_n. Suppose that $(i) = (j)$. Then we have $q^2_{(i)} = p'_{(i),(j)}$, where

$$q_{(i)} : \text{Sk } M_n \to K$$

is the function obtained by taking the Pfaffian of the minor with rows and columns corresponding to the indices in (i). If $(i) \in I_n(r)$, then $q_{(i)} = 0$ if r is odd and $q_{(i)} \neq 0$ if r is even. We see again, as for the case of G_1 above, that f' is a lowest weight vector in $H^\circ(G_2/Q, L'_n)$ and that its weight is

$$-(\epsilon_1 + \ldots + \epsilon_n) = -2\omega_n,$$

where ω_n is the fundamental weight associated to the right end root $\epsilon_{n-1} + \epsilon_n$. Hence the highest weight of $H^\circ(G_2/Q, L'_n)$ is $2i(\omega_n)$. From these considerations, we conclude easily the following:

(i) Let F be the ample generator of Pic G_2/Q. Then $F^2 = L'_n$.

(ii) Let $p'_{(\lambda)} \in H^\circ(G_2/Q, L'_n)$ such that if $((i), (j))$ is the canonical dual pair associated to (λ), then $(i) = (j)$. If $(i) \in I_n(r)$, then $q_{(i)} = 0$ if r is odd, and if r is even, we have a well-determined element $g_{(\lambda)} \in H^\circ(G_2/Q, F)$ such that $g^2_{(\lambda)} = p'_{(\lambda)}$.

(iii) the highest weight of $H^\circ(G_2/Q, F)$ is $i(\omega_n)$ so that Q is the maximal parabolic subgroup associated to the right end root $(\epsilon_{n-1} + \epsilon_n)$.

(iv) Sk M_n is the opposite big cell in G_2/Q.

Note that $i(\omega_n) = \omega_n$ if n is *even* and $i(\omega_n) = \omega_{n-1}$ if n is odd, ω_{n-1} being the fundamental weight associated to the right end root $(\epsilon_{n-1} - \epsilon_n)$.

Let us denote by $W(Q)$ the Weyl group of the maximal parabolic subgroup Q of G_1 (resp. G_2), Q being as above. We claim that

$$W(Q) = W(G_1) \cap W(P_n) \text{ (resp. } W(G_2) \cap W(P_n)).$$

In fact $W(Q)$ is the subgroup of $W(G_1)$ (resp. $W(G_2)$) which fixes the point $e(Q)$ in G_1/Q (resp. G_2/Q), $e(Q)$ being the point associated to the class Q. Now under the canonical immersion

$$G_1/Q \text{ (resp. } G_2/Q) \hookrightarrow H/P_n,$$

$e(Q)$ goes to the point $e(P_n)$ and $W(P_n)$ is the subgroup of $W(H)$ which fixes $e(P_n)$ ($e(P_n)$ is the canonical image of $e_1 \Lambda \overset{n}{\ldots} \Lambda e_n$ in $\mathbf{P}(\Lambda V)$). Thus $W(Q) = W(H) \cap W(G_1)$ (resp. $W(H) \cap W(G_2)$) and we get a canonical inclusion map

$$W(G_1)/W(Q) \hookrightarrow W(H)/W(P_n) \text{ (resp. } W(G_2)/W(Q) \hookrightarrow W(H)/W(P_n)).$$

We have then

Lemma 4.1. (i) Let $w \in W(H)/W(P_n)$ and w_1, w_2 be respectively the *minimal and maximal representatives* in $W(H)$ of w (section 1). Suppose that $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$). Then w_1, w_2 are in $W(G_1)$ (resp. $W(G_2)$) and are in fact, respectively the minimal and maximal representatives in $W(G_1)$ (resp. $W(G_2)$) of w.

(ii) Let us identify $W(H)/W(P_n)$ canonically with $I_{2n}(n)$ (order preserving isomorphism, section 1). Let $(\lambda) \in I_{2n}(n)$ and $((\alpha), (\beta))$ (resp. $((i), (j))$) be the canonical pair (resp. the canonical dual pair) associated to (λ). Then we have

(a) $(\alpha) \in I_n(r)$, $(\beta) \in I_n(r)$ (resp. $(i) \in I_n(s)$, $(j) \in I_n(s)$), $r+s=n$ (the number of indices in (α) and (β) are the same and similarly for (i) and (j)).

(b) $(\lambda) \in W(G_1)/W(Q) \Longleftrightarrow (\alpha) = (\beta)$, (resp. $(i) = (j)$), so that we have a canonical bijection: $W(G_1)/W(Q) \approx \bigcup_{0 \leqslant r \leqslant n} I_n(r)$.

(c) $(\lambda) \in W(G_2)/W(Q) \Longleftrightarrow (\alpha) = (\beta)$, and $(n-r)$ is

even, r being the integer such that $(\alpha) \in I_n(r)$ (resp. $(i) = (j)$ and s is *even* where s is the integer such that $(i) \in I_n(s)$).

(d) $(\lambda) \in W(G_1)/W(Q) \Longrightarrow (j)$ is the complement of (α) in $(1, 2, \ldots, n)$.

(e) the mapping $(\lambda) \longmapsto ((\alpha), (\alpha)), ((\beta), (\beta))$ defines an injection

$$\varphi\colon W(H)/W(P_n) \hookrightarrow W(G_1)/W(Q) \times W(G_1)/W(Q)$$

and the image of φ can be identified with the set of elements (w_1, w_2) such that $m(w_1) = m(w_2)$ (*cf.* Def. 3.1).

(iii) Let $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) and $w = (i_1, \ldots, i_r) \in I_n(r)$ (canonical representation as in (ii) above) Then we have

$$l(w,\ W(G_1)/W(Q)) = \sum_{k=1}^{r} i_k + (n+1)(n-r) - \frac{n(n+1)}{2}$$

$$\text{(resp. } l(w,\ W(G_2)/W(Q)) = \sum_{k=1}^{r} i_k + n(n-r) - \tfrac{1}{2}n(n+1)).$$

(iv) Let $w_1, w_2 \in W(G_1)/W(Q)$ (resp. $W(G_2)/Q$) and

$$w_1 = (i) = (i_1, \ldots, i_r) \in I_n(r) \text{ and } w_2 = (j) = (j_1, \ldots, j_s) \in I_n(s)$$

(representations as in (ii) above). Then

$$w_1 \leqslant w_2 \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)) \Longleftrightarrow (i) \leqslant (j) \text{ (cf. Def. 2.1.)} \Longleftrightarrow w_1 \leqslant w_2 \text{ in } W(H)/W(P_n).$$

(v) Let $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$). Then

$$X(w, G_1/Q) = X(w, H/P_n) \cap G_1/Q$$

$$(\text{resp. } X(w, G_2/Q) = X(w, H/P_n) \cap G_2/Q).$$

Proof: (i) Since w_2 is the maximal representative in $W(H)$ of w, $X(w_2, H/B(H))$ is the inverse image of $X(w, H/P_n)$ under the canonical morphism $H/B(H) \to H/P_n$. We have seen that the involution σ_i $(i = 1, 2)$ on $H/B(H)$ goes down to an involution (denoted by the same σ_i) on H/P_n. Hence the morphism $H/B(H) \to H/P_n$ is equivariant for σ_i. Since $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$), $X(w,H/P_n)$ is stable under the involution under σ_i and it follows that $X(w_2, H/B(H))$ is stable under σ_i. This implies that $w_2 \in W(G_1)$ (resp. $W(G_2)$). We see also that w_2 is a representative in $W(G_1)$ (resp. $W(G_2)$) of $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$). Since w_1 is a minimal representative in $W(H)$ of w, as we mentioned in section 1, we have

$$w_2 = w_1 . w_0(P_n),$$

where $w_0(P_n)$ is the element of largest length in $W(P_n)$. We claim now that $w_0(P_n) \in W(G_1)$ (resp. $W(G_2)$) and is in fact equal to $w_0(Q)$, which is the element of largest length in $W(Q)$. To prove the claim, it suffices to show that $w_0(Q)$, which is afortiori in $W(P_n)$, is of maximal length in $W(P_n)$. This is now an easy consequence of Prop. 3.1. We observe first that $m(w_0(Q)) = 0$ (notation as in Prop. 3.1). Then Prop. 3.1. shows that

$$l(w_0(Q), W(H)) = 2l(w_0(Q), W(G_i)), \; i = 1, 2.$$

One knows that

$$l(w_0(Q), W(G_i)) = \dim Q_i/B(Q_i) \; (i = 1, 2),$$

$$(B(Q_i) = \text{a Borel subgroup in } Q_i).$$

It is easy to check that

$$\dim P_n/B(P_n) = 2 \dim Q_i/B(Q_i) \; (i = 1, 2)$$

$$(B(P_n) = \text{a Borel subgroup in } P_n).$$

It follows then that

$$l(w_0(Q), W(H)) = \dim P_n/B(P_n).$$

This implies that $w_0(Q)$ is the element of maximal length in $W(P_n)$ and the above claim that $w_0(P_n) = w_0(Q)$ follows. Thus we have

$$w_2 = w_1 . w_0(Q).$$

This relation implies in particular that $w_1 \in W(G_1)$ (resp. $W(G_2)$) and is a representative of w. Since w_1, w_2 are respectively the minimal and maximal representatives in $W(H)$ of w, as mentioned in section 1, we have

$$l(w_2, H/B(H)) = l(w_1, H/B(H)) + l(w_0(Q), H/B(H)).$$

It is easy to check that $m(w_1) = m(w_2)$ ($m(w_0(Q)) = 0$ as mentioned above), so that by Prop. 3.1. this formula implies that

$$l(w_2, G_i/B(G_i)) = l(w_1, G_i/B(G_i)) + l(w_0(Q), G_i/B(G_i)), \; i = 1, 2.$$

As we saw in section 1, this relation implies that w_1, w_2 are respectively the minimal and maximal representatives in $W(G_1)$ (resp. $W(G_2)$) of w.

(ii) Let $(\lambda) = (\lambda_1, \ldots, \lambda_n) \in W(H)/W(P_n) = I_{2n}(n)$. A minimal representative w_1 of (λ) in $W(H)$ is given by

$$w_1 = (\lambda_1, \ldots, \lambda_n; \mu_1, \ldots, \mu_n)$$

$(\mu_1, \ldots, \mu_n)$ being the complement of $(\lambda_1, \ldots, \lambda_n)$ in $(1, 2, \ldots, 2n)$, arranged in the increasing order. Thus to write down the condition for (λ) to be in $W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$), on account of (i) above, we have only to express the condition that w_1 is σ-invariant (resp. σ-invariant and $m(w)$ is even, *cf.* Lemma 3.1). The fact that w_1 is σ-invariant is equivalent to saying that

$$\mu_i = 2n+1-\lambda_{n+1-i}, \; 1 \leqslant i \leqslant n.$$

Let now r, $0 \leqslant r \leqslant n$, be such that $\lambda_r \leqslant n$ and $\lambda_{r+1} > n$. Then we see that $(\mu_1, \ldots, \mu_{n-r})$ is the complement of $(\lambda_1, \ldots, \lambda_r)$ in $(1, \ldots, n)$ and that

$$(\lambda_{r+1}, \ldots, \lambda_n) = (2n+1-\mu_{n-r}, \ldots, 2n+1-\mu_1).$$

Now the assertions (ii) follow easily and we leave them as exercise.

(iii) Let $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) and be represented canonically by $w = (i_1, \ldots, i_r) \in I_n(r)$. Let w_1 be a minimal representative of w in $W(H)$. Then we have

$$w_1 = (\underbrace{i_1, \ldots, i_r, 2n+1-j_{n-r}, \ldots, 2n+1-j_1}_{\text{first } n \text{ elements}}, \ldots)$$

where $(j_1, \ldots, j_{n-r})$ is the complement of $(i_1, \ldots, i_r)$ in $(1, \ldots, n)$, arranged in the increasing order (*cf.* (ii) above). Then by (i)

$$l(w_1, W(G_i)/W(Q)) = l(w_1, W(G_i)). \ (i = 1, 2).$$

Further, we have

$$l(w_1, W(H)/W(P_n)) = l(w_1, W(H))$$
$$= \sum_{k=1}^{r} i_k + \sum_{k=1}^{n-r} (2n+1-j_k) - \frac{n(n+1)}{2} \text{ (section 1).}$$

and $m(w_1) = (n-r)$.

Then applying Prop. 3.1, we get

(a) $$l(w, W(G_2)/W(Q)) = l(w, W(G_1)/W(Q)) - m(w), \text{ and}$$
$$l(w, W(G_1)/W(Q)) = \tfrac{1}{2}\, l(w_1, W(H)) + \tfrac{1}{2}\, m(w_1).$$

This gives

(b) $$l(w, W(G_1)/W(Q)) = \tfrac{1}{2}\left\{(n-r)(2n+1) + \sum_{k=1}^{r} i_k - \left(\frac{n(n+1)}{2} - \sum_{k=1}^{r} i_k\right) - \frac{n(n+1)}{2}\right\} + \tfrac{1}{2}(n-r) = \left(\sum_{k=1}^{r} i_k\right) + (n+1)(n-r) - \frac{n(n+1)}{2}.$$

Now (a) and (b) together prove (iii).

(iv) We have seen that $f = p_{(1, \ldots, n)}$ is a lowest weight vector in $H^\circ(H/P_n, L_n)$. Then as we saw in section 1 (towards the end), the family

$$\{p_\tau\},\ \tau \in W(H)/W(P_n),\ p_\tau \text{ being defined as } p_\tau = \tau.f, \text{ has the property:}$$

(*) $$p_{\tau_1} \mid X(\tau_2, H/P_n) \neq 0 \iff \tau_1 \leqslant \tau_2 \text{ in } W(H)/W(P_n).$$

We have seen that the restriction of f to G_1/Q is a lowest weight vector in $H^\circ(H/P'_n, L'_n)$ $L'_n = L_n$ restricted to G_1/Q. Further, we have an element $g \in H^\circ(G_2/Q, F)$, $F^2 = L'_n$, $L'_n = L_n$ restricted to G_2/Q, such that $g^2 = f$ restricted to G_2/Q and g is a lowest weight vector in $H^\circ(G_2/Q, F)$. Let us denote by g_0 the restriction of f to G_1/Q (resp. $g = g_0$, the element $g \in H^0(G_2/Q, F)$ as above). Then the family

$$\{g_\tau\},\ g_\tau = \tau.g_0,\ \tau \in W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q))$$

has the property (*cf.* section 1)

$$\tau_1 \leqslant \tau_2 \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)) \iff g_{\tau_1} \mid X(\tau_2, G_1/Q) \neq 0$$
$$\text{(resp. } \mid g_{\tau_1} X(\tau_2, G_2/Q) \neq 0).$$

Then from (*), we conclude that

$$\tau_1 \leqslant \tau_2 \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)) \Longleftrightarrow \tau_1 \leqslant \tau_2 \text{ in } W(H)/W(P_n)$$

since, because of $X(\tau_2, G_1/Q)$ (resp. $X(\tau_2, G_2/Q)) \subset X(\tau_2, H/P_n)$, one has

$$\left.\begin{matrix} g_{\tau_1} \mid X(\tau_2, G_1/Q) \neq 0 \\ \text{(resp. } g_{\tau_1} \mid X(\tau_2, G_2/Q \neq 0) \end{matrix}\right\} \Longrightarrow f_{\tau_1} \mid X(\tau_2, H/P_n) \neq 0$$

Thus to conclude the proof of (iv), we have only to prove the following implication:

$w_1, w_2 \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$ and $w_1 \neq w_2$, then

$$w_1 \leqslant w_2 \text{ in } W(H)/W(P_n) \Longrightarrow w_1 \leqslant w_2 \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)).$$

We shall now show that we can find $w'_1 \in W(G_1)/W(Q)$ (resp. $W(G)_2/W(Q)$) such that

(a) $w_1 \leqslant w_1' \leqslant w_2$ in $W(H)/W(P_n)$.

(b) $l(w_1', W(G_1)/W(Q))$ (resp. $l(w_1', W(G_2)/W(Q))$

$= l(w_2, W(G_1)/W(Q)) - 1$ (resp. $l(w_2, W(G_2)/W(Q)) - 1$)

(c) $w_1' = w_2 r_\alpha$, where r_α is the reflection with respect to a root $\alpha \in R^+(G_1)$ (resp. $R^+(G_2)$ (in this identity, we take for w_1' and w_2 their respective minimal representatives in $W(H)$). These assertions would imply that

$$w_1' \leqslant w_2 \text{ in } W(G_1) \text{ (resp. } W(G_2))$$

and thus to complete the proof of (iv), it suffices to prove the above assertions. We set

$$w_1 = (i_1, \ldots, i_r) \in I_n(r), w_2 = (j_1, \ldots, j_s) \in I_n(s).$$

We have $r \geqslant s$ since $w_1 \leqslant w_2$ in $W(H)/W(P_n)$ is equivalent to $(i) \leqslant (j)$ in the sense of Definition 2.1.

Case I. Suppose that

$$i_1 = j_1, i_2 = j_2, \ldots, i_{k-1} = j_k - 1 \text{ and } i_k \neq j_k, k \leqslant s.$$

Then we set

$$w_1' = (j_1, \ldots, j_{k-1}, j_k - 1, j_{k+1}, \ldots, j_s).$$

We observe that

$$(j_k - 1) \notin (j_1, \ldots, j_s).$$

Case II. We have

$$i_1 = j_1,\ i_2 = j_2,\ \ldots,\ i_s = j_s.$$

Then we have $r > s$ since otherwise $w_1 = w_2$. Then we set

Case G_1: $w_1' = (j_1, \ldots, j_s, n)$

Case G_2: $w_1' = (j_1, \ldots, j_s, n-1, n)$ (note that in this case $r-s \geqslant 2$).

The required property (a) above is immediate. The property (b) above also follows immediately from (iii) of Lemma 4.1, proved above. It remains to prove (c) to conclude the proof of (iv).

Case I (as above): Let

$w_2 = (b_1, \ldots, b_{2n})$—minimal representative of w_2. Let p be the integer such that

$$b_{2n+1-p} = (j_k - 1).$$

Since $(j_k - 1) \notin (j_1, \ldots, j_s)$, we see that $(s+1) \leqslant p \leqslant n$. Let r_α denote the reflection (in $W(G_1)$, resp. $W(G_2)$) with respect to the positive root $\alpha = \epsilon_k + \epsilon_p$, which is in $R^+(G_1)$ as well as $R^+(G_2)$, identified as subsets of $X(T_{2n})$ (*cf.* section 3). We see that r_α as an element of $W(H)$ is the product of transpositions

$$r_\alpha = (k, 2n+1-p)\,(p, 2n+1-k) \text{ (note } k \leqslant s,\ s+1 \leqslant p).$$

It is now a simple exercise to deduce that

$$w_1' = w_2\, r_\alpha.$$

Case II (as above): We set

Case G_1: r_α = reflection with respect to $\alpha = 2\epsilon_{s+1} \in R^+(G_1)$ (note $s+1 \leqslant n$).

Case G_2: r_α = reflection with respect to $\alpha = \epsilon_{s+1} + \epsilon_{s+2} \in R^+(G_2)$ (note $s+2 \leqslant n$).

As an element of $W(H)$, we observe that

Case G_1: r_α = transposition $(s+1, 2n-s)$.

Case G_2: r_α = product of transpositions $(s+1, 2n-s-1)\,(s+2, 2n-s)$. It is now a simple exercise to deduce that $w_1' = w_2\, r_\alpha$.

This concludes the proof of (iv) of Lemma 4.1.

(v) This is an immediate consequence of Lemma 3.2 and (iv) of Lemma 4.1. In fact for $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$), set

$$Y = X(w, H/P_n) \cap G_1/Q \text{ (resp. } X(w, H/P_n) \cap G_2/Q).$$

We see that Y is a union of Schubert varieties in G_1/Q (resp. G_2/Q) and by Lemma 3.2, it follows $X(w,G_1/Q)$ (resp. $X(w,G_2/Q)$) is an irreducible component of Y. Suppose that there is *another* irreducible component of Y, then it is a Schubert variety in G_1/Q (resp. G_2/Q) and it is of the form $X(w', G_1/Q)$ (resp. $X(w', G_2/Q')$), $w' \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) with $w' \not\leq w$. We have, a fortiori,

$$w' \not\leq w \text{ in } W(H)/W(P_n)$$

so that by (iv) of Lemma 4.1, it follows that

$$w' \not\leq w \text{ in } W(G_1)/W(Q) \text{ (resp. } W(G_2)/W(Q)).$$

This leads to a contradiction. This proves (v) and the proof of Lemma 4.1 is complete.

The basic results in section 4 can now be summarized in the following:

Theorem 4.1. Let Q denote the maximal parabolic subgroup in $G_1 = Sp(2n)$ (resp. $G_2 = SO(2n)$) associated to the right end root α_n in the Dynkin diagram (notations as in Bourbaki [2]). Then we have the following:

(i) We have a canonical imbedding of G_1/Q (resp. G_2/Q) in H/P_n (Grassmannian of n planes in a $2n$ dimensional vector space) such that the *opposite big cell* in H/P_n restricts to the opposite big cell in G_1/Q (resp. G_2/Q); further if we take the canonical identification of the opposite big cell in H/P_n with M_n = Space of $(n \times n)$ matrices (*cf.* Th. 2.1. and section 2), the opposite big cell in G_1/Q (resp. G_2/Q) can be identified with Sym M_n = Space of $(n \times n)$ symmetric matrices (resp. Sk M_n = $(n \times n)$ skew symmetric matrices)

(ii) Let $W(H)$, $W(G_1)$,.. etc. denote the Weyl groups of H, G_1, ..., etc. Then we have canonical inclusions

(a) $W(G_1)$ (resp. $W(G_2)$) $\hookrightarrow W(H)$

(b) $W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) $\hookrightarrow W(H)/W(P_n)$.

The inclusion (b) preserves the partial orders in each. For $(\lambda) \in I_{2n}(n) \approx W(H)/W(P_n)$, let $((\alpha), (\beta))$ (resp. $((i), (j))$) denote the canonical (resp. canonical dual) pair associated to (λ) (*cf.* Def. 2.2). Then $(\lambda) \in W(G_1)/W(Q)$ (resp. $WG_2)/W(Q)$) if and only if $(\alpha) = (\beta)$ (resp. $(\alpha) \neq (\beta)$ and the number of elements in (i) or (j) is even).

(iii) Let L_n denote the ample generator of Pic H/P_n and L'_n the restriction of L_n to G_1/Q (resp. G_2/Q). Then L'_n is the ample generator of Pic G_1/Q (resp. $L'_n = F^2$, where F is the ample generator of Pic G_2/Q). Let $f = p_{(1,\ldots,r)}$ be a lowest weight vector in $H^0(H/P_n, L_n)$. Then the restriction f' of f to G_1/Q (resp. G_2/Q) is a lowest

weight vector in $H^\circ(G_1/Q, L_n')$ (resp. $f' = g^2$, where g is a lowest weight vector in $H^\circ(G_2/Q,F)$). If $\{p_\tau\}$, $\tau \in W(H)/W(P_n)$, is as usual the family defined by $p_\tau = \tau.f$, then the restriction of p_τ (or to be more precise p_τ/f) to Sym M_n, identified as the opposite big cell in G_1/Q, is the function $p'_{(i),(j)}$: Sym $M_n \to K$, defined as follows: $p'_{(i),(j)}(Z)$ = determinant of the minor of Z with rows and columns associated to the indices of (i), (j) respectively, $Z \in$ Sym M_n.

$((i), (j))$ = canonical dual pair associated to τ.

Further the restriction of p_τ to Sk M_n, identified as the opposite big cell in G_2/Q, is the function $q^2_{(i)}$: Sk $M_n \to K$, defined as follows:

$q_{(i)}(Z)$ = Pfaffian of the square minor of Z with rows associated to the indices of (i), $Z \in$ Sk M_n,

$((i), (j))$ = the canonical dual pair associated to τ.

Remark 4.1. (1) We see that the big cell in H/P_n restricts to the big cell in G_1/Q. This is also the case for G_2/Q, if n is even; however if n is odd, the restriction of the big cell in H/P_n to G_2/Q is *empty*.

(2) We have seen (*cf.* Th. 4.1) that the partial order in $W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) is induced from $W(H)/W(P_n)$. It is also true that the partial order in $W(G_1)$ is induced from $W(H)$; however this is not the case for $W(G_2)$.

5. Interpretation of the standard monomial theory of De Concini and Procesi [5] on the space of symmetric and skew symmetric matrices

We keep the basic notations of the previous sections.

Definition 5.1. (i) Let D_k denote the determinantal variety in M_n as in Def. 2.3. Set

$$D_k(G_1) \text{ (resp. } D_k(G_2)) = D_k \cap \text{Sym } M_n \text{ (resp. } D_k \cap \text{Sk } M_n \text{ and } k \text{ is even)}.$$

We call these *determinantal varieties in* Sym M_n (*resp.* Sk M_n). We observe that if $w \in W(G_1)/W(Q)$ (resp. $W(G_2)/W(Q)$) is such that the canonical *dual* pair associated to w is $((1, \ldots, k)\ (1, \ldots, k))$, then $D_k(G_1)$ resp. $(D_k\ (G_2))$ is the *opposite big cell* in $X(w, G_1/Q)$ (resp. $X(w, G_2/Q)$), in particular we see that $D_k(G_1)$ (resp. $D_k(G_2)$) is irreducible.

(ii) Let A_n denote the coordinate ring of Sym M_n. For a pair $((i), (j))$, $(i) \in I_n(r)$, $(j) \in I_n(r)$, $0 \leqslant r \leqslant n$, let $p'_{(i),(j)}$: Sym $M_n \to K$ denote the function defined in section 4 (*cf.* iii, Th. 4.1), so that $p'_{(i),(j)} \in A_n$. We call a *standard monomial in* A_n, an

expression of the following type:

$$p'_{(i),(j)}\, p'_{(i)',(j)'}\, p'_{(i)'',(j)''}\, p'_{(i)''',(j)'''} \cdots$$

$$(i) \leqslant (j) \leqslant (i)' \leqslant (j)' \leqslant (i)'' \leqslant (j)'' \leqslant \ldots \text{ (}cf.\text{ Def. 2.1)}$$

and the number of $p_{(i),(j)}$ in this expression is called the *length* of this standard monomial.

(iii) Let B_n denote the coordinate ring of Sk M_n. For (i) $\in I_n(r)$, $0 \leqslant r \leqslant n$, let $q_{(i)}$: Sk $M_n \to K$ denote the function defined in section 4 (*cf.* (iii), Th. 4.1.), so that $q_{(i)} \in B_n$. We call a *standard monomial in* B_n, an expression of the following type:

$$q_{(i)'}\, q_{(i)''}\, q_{(i)'''} \cdots$$

$$(i)' \leqslant (i)'' \leqslant (i)''' \leqslant \ldots \text{ (}cf.\text{ Def. 2.1).}$$

Theorem 5.1. (i) Let $w \in W(G_2)/W(Q)$. Then the Schubert variety $X(w, G_2/Q)$ in G_2/Q is normal (in fact, it is also Cohen-Macaulay). In particular, the determinantal varieties $D_k(G_2)$ are normal.

(ii) Let g be a lowest weight vector in $H^\circ(G_2/Q, F)$, $F =$ the ample generator of Pic G_2/Q. Set

$$q_\tau = \tau.g,\ \tau \in W(G_2)/W(Q).$$

We call a *standard monomial of length m* on G_2/Q, an element in $H^\circ(G_2/Q, F^m)$ of the form:

$$q_{\tau_1} q_{\tau_2} \cdots q_{\tau_m},\ \tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m.$$

Then distinct standard monomials of length m of the form

$$q_{\tau_1} q_{\tau_2} \cdots q_{\tau_m},\ \tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m,\ w \geqslant \tau_1$$

form a basis of the vector space $H^\circ(X(w, G_2/Q), F^m)$.

(iii) Distinct standard monomials in B_n of the form

$$q_{(i)'}\, q_{(i)''}\, q_{(i)'''} \cdots$$

$(1, \ldots, k) \leqslant (i)' \leqslant (i)'' \leqslant \ldots$ (i.e. the number of elements in $(i)'$, $(i)''$ is $\leqslant k$) form a basis of the coordinate ring of $D_k(G_2)$ (as a vector space).

Proof. We note that the fundamental weight ω_n is *minuscule*, in the sense that if M is the irreducible representation with highest weight ω_n, when the base field is of

characteristic zero, then all the weights in M are of the form $\tau(\omega_n)$, $\tau \in W(G_2)$ (Bourbaki [2]; Seshadri [22]). Then the assertions (i) and (ii) are just special cases of Seshadri [22] (note that in Seshadri [22] the weight vectors in $H^\circ(G_1/Q, F)$ are indexed as $\tau g'$, g' a highest weight vector), where the Hodge-Young theory of standard monomials is generalized to G/P, G a semi-simple group and P is a maximal parabolic subgroup associated to a minuscule weight. Then the assertion (iii) follows from (ii), in the same way as we had done in the proof of Th. 2.1. for the case H/P_r.

Definition 5.2. Let f' be a lowest weight vector in $H^\circ(G_1/Q, L'_n)$, where L'_n is the ample generator of Pic G_1/Q. Set

$$\{q_\tau\},\ \tau \in W(G_1)/W(Q),\ q_\tau = \tau \cdot f'.$$

Recall that in section 4, for $\tau \in W(H)/W(P_n)$, we had defined p'_τ = restriction of $p_\tau \in H^\circ(H/P_n, L_n)$ ($p_\tau = \tau \cdot f$, f = Plücker coordinate $p_{(1,\ldots,n)}$). Hence $q_\tau = p'_\tau$, when $\tau \in W(G_1)/W(Q)$. For $\tau \in W(H)/W(P_n)$, let (τ_1, τ_2) be the pair of elements in $W(G_1)/W(Q)$, which is the image of τ in the canonical injection.

$$W(H)/W(P_n) \hookrightarrow W(G_1)/W(Q) \times W(G_1)/W(Q) \quad (cf.\ (e),\ \text{(ii)}\ \text{of Lemma 4.1}).$$

Let us call τ *admissible*, if $\tau_1 \geqslant \tau_2$. By Lemma 4.1, we see easily that given a pair of elements $(\tau_1, \tau_2) \in W(G_1)/W(Q)$, (τ_1, τ_2) comes from an admissible element τ if and only if

$$(*) \qquad \begin{cases} \text{(i)}\ \tau_1 \geqslant \tau_2 \text{ in } W(G_1)/W(Q), \\ \text{(ii)}\ m(\tau_1) = m(\tau_2)\ (cf.\ \text{Def. 3.1}). \end{cases}$$

Thus the set of admissible elements can be canonically identified with the set of pairs (τ_1, τ_2) in $W(G_1)/W(Q)$, satisfying (i) and (ii) of (*) and we call such a pair (τ_1, τ_2) an *admissible pair* in $W(G_1)/W(Q)$ and the element $\tau \in W(H)/W(P_n)$, the *admissible element in* $W(H)/W(P_n)$ associated to τ_1, τ_2. Let (τ_1, τ_2) be an admissible pair in $W(G_1)/W(Q)$ and τ the associated admissible element in $W(H)/W(P_n)$. We set

$$q_{\tau_1, \tau_2} = p'_\tau.$$

Let $(\alpha_1\ \alpha_2)$, (β_1, β_2) be two pairs of admissible elements in $W(G_1)/W(Q)$ and α, β respectively the associated admissible elements in $W(H)/W(P_n)$. Then we define $(\alpha_1, \alpha_2) \geqslant (\beta_1, \beta_2)$ by

$$(\alpha_1, \alpha_2) \geqslant (\beta_1, \beta_2) \Longleftrightarrow \alpha_2 \geqslant \beta_1\ (\Longleftrightarrow \alpha \geqslant \beta).$$

We can identify $W(G_1)/W(Q)$ with pairs of (admissible) elements of the form (τ, τ), $\tau \in W(G_1)/W(Q)$ and the order in $W(G_1)/W(Q)$ is consistent with the order introduced above. If (τ_1, τ_2) is an admissible pair in $W(G_1)/W(Q)$, we call it a *non-trivial pair* (resp. *trivial pair*) if $\tau_1 \neq \tau_2$ (resp. $\tau_1 = \tau_2$). We now observe

$$q_{\tau, \tau} = q_\tau.$$

We call a *standard monomial of length m on* G_1/Q, an element in $H^\circ(G_1/Q, (L'_n)^m)$ of the form

$$q_{\alpha_1, \alpha_2} q_{\beta_1, \beta_2} q_{\gamma_1, \gamma_2} \cdots,$$

$(\alpha_1, \alpha_2) \geqslant (\beta_1, \beta_2) \geqslant \ldots$; (α_1, α_2), (β_1, β_2), ... being admissible pairs in $W(G_1)/W(Q)$.

For $\tau \in W(H)/W(P_n)$, let $p'_{(i),(j)}$ be the function

$$p'_{(i),(j)} : \text{Sym } M_n \to K$$

which is the restriction of p'_τ to Sym M_n. Then we see that

$$(i) \leqslant (j) \iff \tau \text{ is admissible.}$$

Then we have

$$q_{\tau_1, \tau_2} \mid \text{Sym } M_n = p'_{(i),(j)},$$

(τ_1, τ_2) being essentially the *canonical pair* associated to τ and $((i), (j))$ the canonical dual pair associated to τ. We see that the restriction to Sym M_n of a standard monomial of length m on G_1/Q is standard in the sense of (ii), Def. 5.1 and conversely if the restriction of

$$p'_{\theta_1} \cdots p'_{\theta_m}, \quad \theta_i \in W(H)/W(P_n)$$

to Sym M_n is standard, then it is standard on G_1/Q.

Theorem 5.2. Let $w_k \in W(G_1)/W(Q)$ be the element such that the canonical dual pair associated to w_k is $((1, \ldots, k), (1, \ldots k))$ $(0 \leqslant k \leqslant n)$. Then we have:

(i) Distinct standard monomials of length m of the form (*cf.* Def. 5.2)

$$q_{\tau_1, \tau'_1} \cdots q_{\tau_m, \tau'_m}\ ;\ w_k \geqslant \tau_1$$

form a basis of $H^\circ(X, (w_k, G_1/Q), (L'_n)^m)$ (to be precise, we take the restrictions to $X(w_k, G_1/Q)$ of these elements in $H^\circ(G_1/Q, (L'_n)^m)$.

(ii) Distinct standard monomials in A_n (*cf.* Def. 5.1) of the form

$$p'_{(i),(j)}\ p'_{(i)',(j)'}\ p'_{(i)'',(j)''} \cdots$$

$(1, \ldots k) \leqslant (i)$ i.e. the number of elements in (i), (j), $(i)'$, $(j)'$ etc. is $\leqslant k$.

form a basis of the coordinate ring (considered a vector space) of the determinantal variety $D_k(G_1)$ in Sym M_n.

Proof. The assertion (ii) is proved in De Concini-Procesi [5]. Since $D_k(G_1)$ is the opposite big cell in $X(w_k, G_1/Q)$, the assertion (i) follows from (ii), in fact as in the proof of Th. 2.1 and Remark 2.1, we see that the assertions (i) and (ii) are equivalent.

Remark 5.1. (i) The assertion (i) of Th. 5.2 does not follow from (Seshadri [22]) as the fundamental weight ω_n is *not* minuscule (we are in type C_n).

(ii) It can be shown that the varieties $D_k(G_1)$ are normal and Cohen-Macaulay. This is proved in Kutz [14] and from this one can deduce that $X(w_k, G_1/Q)$ is normal and Cohen-Macaulay.

We shall now give an intrinsic description of admissible pairs in $W(G_1)/W(Q)$, which will suggest how the theory of standard monomials developed in (Seshadri [22]) for G/P when P is minuscule, could be generalized to the nonminuscule case.

Suppose now that G is a semi-simple algebraic group for which we fix a maximal torus T and a Borel subgroup B, $T \subset B$. Let P be a maximal parabolic subgroup in G, $P \supset B$. Let $Ch(G/P)$ (resp. $Ch(G/B)$) denote the Chow ring of G/P (resp. G/B). If X is a Schubert variety in G/P (resp. G/B), we denote by $[X]$, the class in $Ch(G/P)$ (resp. $Ch(G/B)$) represented by X. One knows that $Ch(G/P)$ (resp. $Ch(G/B)$) is a free $\mathbf{Z}$-module and that the classes represented by distinct Schubert varieties form a $\mathbf{Z}$-basis of $Ch(G/P)$ (resp. $Ch(G/B)$). Hence if X, Y are Schubert varieties in G/P (resp. G/B), the intersection product $[X] \cdot [Y]$ can be expressed as

$$[X] \cdot [Y] = \Sigma_Z \, d_Z \, [Z]$$

where Z runs through the set of all Schubert varieties in G/P (resp. G/B). We call d_z the intersection multiplicity of Z in $[X] \cdot [Y]$ (or simply $X \cdot Y$).

We denote by $[H]$ the element in $Ch(G/P)$, represented by the unique Schubert variety H of codimension one in G/P. This is the class determined by a hyperplane intersection of G/P in its canonical projective imbedding. We denote by the same $[H]$ the element in $Ch(G/B)$, determined by the Schubert variety of codimension one in G/B, which is the inverse image of H by the canonical morphism $G/B \to G/P$.

Let $w_1, w_2 \in W(G)$ (Weyl group of G) such that we have

(i) $X(w_1, G/B) \subset X(w_2, G/B)$ i.e. $w_1 \leqslant w_2$ in $W(G)$

(ii) $X(w_1, G/B)$ is of codimension one in $X(w_2, G/B)$. Then one knows that $w_1 = w_2 r_\alpha$ (see for example Demazure [6])

where r_α is the reflection with respect to a positive root α (with reference to T, B). Then Chevalley's formula (Chevalley [4]; Demazure [6]) states that the intersection multiplicity of $X(w_1, G/B)$ in $[X(w_2, G/B)] \cdot [H]$ is $\langle \omega, \alpha^V \rangle$, where ω is the fundamental weight to which P is associated. Suppose now that $u, v \in W(G)/W(P)$ are such that

(i) $X(u, G/P) \subset X(v, G/P)$ i.e. $u \leqslant v$ in $W(G)/W(P)$.

(ii) $X(u, G/P)$ is of codimension one in $X(v, G/P)$.

Let w_1, w_2 be respectively *minimal representatives* in $W(G)$ of u, v. Then again we have $w_1=w_2 r_\alpha$, where r_α is the reflection with respect to a positive root α. From the preceding Chevalley's formula, it follows immediately (using the projection formula) that the intersection multiplicity of $X(u,G/P)$ in $[X(v, G/P)] \cdot [H]$ is $\langle \omega, \alpha^v \rangle$.

Now the crucial result is

Lemma 5.1: Let v, u be a pair of elements in $W(G_1)/W(Q)$ such that $u \leqslant v$ and $X(u,G_1/Q)$ is of codimension one in $X(v, G_1/Q)$. Then (v, u) is an *admissible pair* if and only if the intersection multiplicity of $X(u, G_1/Q)$ in $[X(v, G_2/Q)] \cdot [H]$ is 2.

Proof. Let w_1, w_2 be the minimal representatives in $W(G_1)$ of u and v respectively. Then $w_1=w_2 r_\alpha$, where r_α is the reflection with respect to a positive root α.

Suppose now that (v, u) is an *admissible pair.* Then since $X(u, G_1/Q)$ is of codimension one in $X(v, G_1/Q)$, we see by (iii) of Lemma 4.1. that u and v are of the following form.

$$u=(j_1, j_2, \ldots, j_{k-l}, j_k{}^{-1}, j_{k+1} \ldots, j_s), \quad u \in I_n(s), \; v \in I_n(s), \; s \leqslant n.$$

$$v = (j_1, j_2, \ldots, j_{k-1}, j_k, j_{k+1}, \ldots, j_s), \quad (j_k - 1) \notin (j_1, \ldots, j_s)$$

(u and v differ only in the kth index and the kth index of u is one less than that of v).

Let $w_2 = (b_1, \ldots, b_{2n}) \in W(G_1)$ and p be the integer such that

$$b_{2n+1-p} = (j_k - 1)$$

Then as in the proof of (iv), Lemma 4.1, we see that $s+1 \leqslant p \leqslant n$ and that α is the positive root

$$\alpha = \epsilon_k + \epsilon_p \in R^+(G_1).$$

Then by Chevalley's formula (mentioned above), it follows that the intersection multiplicity of $X(u, G_1/Q)$ in $[X(v, G_1/Q)] \cdot [H]$ is given by $\langle \omega, (\epsilon_k + \epsilon_p)^V \rangle$. We have

$$\langle \omega, (\epsilon_k + \epsilon_p)^V \rangle = \frac{2(\omega, \epsilon_k + \epsilon_p)}{(\epsilon_k + \epsilon_p, \epsilon_k + \epsilon_p)} = (\omega, \epsilon_k + \epsilon_p) = 2$$

$$(\omega = \epsilon_1 + \ldots + \epsilon_n).$$

Suppose on the other hand that the intersection multiplicity of $X(u, (G_1/Q)$ in $[(X(v, G_1/Q)] \cdot [H]$ is 2. Then to prove that (v, u) is an admissible pair, we ha

only to show that

$$m(u) = m(v) \text{ i.e. } m(w_1) = m(w_2) \text{ (cf. Def. 5.2).}$$

Let

$$v = (i_1, \ldots, i_s) \in I_n(s),\ s \leqslant n.$$

Suppose that $m(u) \neq m(v)$. Then the hypothesis that $u \leqslant v$ and $X(u, G_1/Q)$ is o codimension one in $X(v, G_1/Q)$ implies easily (by (iii) of Lemma 4.1) that

$$u = (i_1, \ldots i_s, n)$$

i.e. $m(u) = m(v)+1 = s+1$, the first s elements of u and v are the same and the las element of u is n. Then again as we saw in the proof of (iv), Lemma 4.1, α is th positive root $2\epsilon_{s+1}$. Then by Chevalley's formula, the intersection multiplicity o $X(u, G_1/Q)$ in $[X(v, G_2/Q)] \cdot [H]$ is given by $\langle \omega, (2\epsilon_{s+1})^V \rangle$. We have

$$\langle \omega, (2\epsilon_{s+1})^V \rangle = \frac{2(\omega, 2\epsilon_{s+1})}{(2\epsilon_{s+1}, 2\epsilon_{s+1})} = (\omega, \epsilon_{s+1}) = 1.$$
$$(\omega = \epsilon_1 + \ldots . \epsilon_n).$$

This leads to a contradiction and hence $m(u) = m(v)$. This completes the proof c Lemma 5.1.

Corollary: Let (τ, φ) be a pair of elements in $W(G_1)/W(G_1)$. Then it is a *non-trivi admissible pair* if and only if we can find a chain of elements $\{\tau_i\}, 1 \leqslant i \leqslant m,$ $\in W(G_1)/W(Q)$ having the following properties:

(i) $\tau_1 = \tau,\ \tau_m = \varphi$ and $\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m$

(ii) $X(\tau_i, G_1/Q)$ is of codimension one in $X(\tau_{i-1}, G_1/Q)$ and the intersectio multiplicity of $X(\tau_i, G_1/Q)$ in $[X(\tau_{i-1} G_1/Q)] \cdot [H]$ is 2 $(2 \leqslant i \leqslant m)$.

Proof. If the conditions (i) and (ii) are satisfied, it is clear by the above lemma th (τ, φ) is an admissible pair, since it follows that

$$m(\tau_1) = m(\tau_2) = \ldots = m(\tau_m).$$

Suppose then that on the other hand (τ, φ) is an admissible pair in $W(G_1)/W(Q$ Then we have

$$\left.\begin{array}{l} \tau = (j_1, \ldots, j_s),\ \tau \in I_n(s) \\ \varphi = (i_1, \ldots, i_s),\ \varphi \in I_n(s) \end{array}\right\} 0 \leqslant s \leqslant n.$$

Suppose that we have

$$i_1 = j_1,\ i_2 = j_2, \ldots, i_{k-1} = j_{k-1} \text{ and } i_k < j_k,\ k \leqslant s.$$

Then we set

$$\varphi' = (j_1, \ldots, j_{k-1}, j_k - 1, j_{k+1} \ldots, j_s).$$

Then we have

(a) $\tau \geqslant \varphi' \geqslant \varphi$.

(b) (τ, φ') and (φ', φ) are admissible pairs.

(c) $X(\varphi', G_1/Q)$ is of codimension one in $X(\tau, G_1/Q)$.

Then by Lemma 5.1 we construct by an easy inductive argument, a chain which satisfies the properties (i) and (ii) above. This completes the proof of the corollary.

Remark 5.1. It can be checked that the elements in $H^\circ(G_1/Q, L'_n)$

$\{q_{\tau_1, \tau_2}\}$, (τ_1, τ_2) admissible pair in $W(G)/W(Q)$ have the following properties:

(i) The restriction of q_{τ_1, τ_2} to $X(w, G_1/Q)$, $w \in W(G_1)/W(Q)$ is not identically zero if and only if

$$w \geqslant \tau_1, \text{ i.e., } (w, w) \geqslant (\tau_1, \tau_2).$$

(ii) the weight of q_{τ_1, τ_2} is

$$\tfrac{1}{2}((\tau_1(\omega') + \tau_2(\omega')) = -\tfrac{1}{2}(\tau_1(\omega_n) + \tau_2(\omega_n)).$$

ω' being the lowest weight of the G_1-module $H^\circ(G_1/Q, L_n')$ (note that $\omega' = \omega_0(i(\omega_n)) = -\omega_n$).

6. The basic conjectures

The results of section 5 as well as computations in low rank suggest the following conjectures:

Let G be a semi-simple, simply-connected, algebraic group defined over an algebraically closed field K. Fix a maximal torus T and a Borel subgroup B, $T \subset B$. We refer to roots, Bruhat decomposition etc. with respect to this choice of T, B. Let P be a *maximal parabolic subgroup*, $P \supset B$, associated to a fundamental weight ω. We say that P or ω is of *classical type* if

$$\langle \omega, \alpha^\vee \rangle = 0, \pm 1 \text{ or } \pm 2, \forall \text{ root } \alpha \text{ of } G.$$

The hypothesis that P is of classical type implies (on account of Chevalley's formula, section 5, also Chevalley [4]; Demazure [6] that if X, Y are Schubert

varieties in G/P such that Y is of codimension one in X and H is the codimension one Schubert variety in G/P, then the intersection multiplicity of Y in $(X)\cdot(H)$ is utmost 2 (*cf* section 5 for the notations $X, \ldots$, etc.). In fact it can be seen that this property is an equivalent description of P to be of classical type.

Let $W(G)$ (resp. $W(P)$) denote the Weyl group of G (resp. P). We denote by $X(w, G/B)$ (resp. $X(w, G/P)$) the Schubert variety in G/B (resp. G/P) associated to $w \in W(G)$ (resp. $W(G)/W(P)$).

Definition 6.1. (i) Let (τ, φ) be a pair of elements in $W(G)/W(P)$. We call it an *admissible pair* if

(a) Either $\tau = \varphi$, in which case we call it a *trivial admissible pair*, or

(b) we can find $\tau_1, \ldots, \tau_m \in W(G)/W(P)$ such that

(1) $X(\tau) = X(\tau_1) \supset X(\tau_2) \supset \ldots \supset X(\tau_m) = X(\varphi)$

i.e. $\tau_1 = \tau \geqslant \tau_2 \geqslant \ldots \geqslant \tau_m = \varphi$,

(2) $X(\tau_i)$ is of codimension one in $X(\tau_{i-1})$ and the intersection multiplicity of

$X(\tau_i)$ in $[X(\tau_{i-1})] \cdot [H]$ is 2 $(2 \leqslant i \leqslant m)$.

In case (b), we say that (τ, φ) is a *non-trivial admissible pair*.

(ii) Let (τ_1, φ_1), (τ_2, φ_2) be two admissible pairs in $W(G)/W(P)$. Then we define

$(\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2)$ if $\varphi_1 \geqslant \tau_2$ in $W(G)/W(P)$.

(Note that this relation $\geqslant$ is *not* a partial order in the set of admissible pairs in $W(G)/W(P)$. It satisfies the axiom of transitivity but not of reflexivity).

Conjectures I. Let G be as above and suppose that P is a maximal parabolic subgroup $(P \supset B)$ of classical type. Let ω be the fundamental weight associated to P. Let L be the ample generator of Pic G/P. Then given an *admissible pair* (τ, φ) in $W(G)/W(P)$, we can find a $p_{\tau, \varphi} \in H^0(G/P, L)$ having the following properties:

(i) $p_{\tau, \varphi}$ does not vanish on the Schubert variety $X(w, G/P)$, $w \in W(G)/W(P)$, if and only if

$$w \geqslant \tau \text{ i.e. } (w, w) \geqslant (\tau, \varphi)$$

(ii) the one dimensional space spanned by $p_{\tau, \varphi}$ is stable under T and the weight of $p_{\tau, \varphi}$ is

$$-\tfrac{1}{2}(\tau(\omega) + \varphi(\omega))$$

(iii) let us call an element of $H^0(G/P, L^m)$ of the form

$$p_{\tau_1, \varphi_1} p_{\tau_2, \varphi_2} \cdots p_{\tau_m, \varphi_m},$$

$$(\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2) \geqslant \cdots \geqslant (\tau_m, \varphi_m)$$

a *standard monomial of length m.* Then distinct standard monomials of length m form a basis of $H^\circ(G/P, L^m)$, $m \geqslant 1$. In fact distinct standard monomials of length m as above such that $w \geqslant \tau_1$, form a basis of $H^\circ(X(w, G/P), L^m)$, $m \geqslant 1$.

Let us call a *Young tableau* (or *diagram*) *of length m* in $W(G)/W(P)$, a sequence of m admissible pairs in $W(G)/W(P)$ of the following type

$$\begin{cases} (\tau_1, \varphi_1), (\tau_2, \varphi_2), \ldots, (\tau_m, \varphi_m), \\ (\tau_1, \varphi_1) \geqslant (\tau_2, \varphi_2) \geqslant \cdots \geqslant (\tau_m, \varphi_m). \end{cases}$$

The above conjectures imply in particular that

$$\dim H^\circ(G/P, L^m) = \#\{\text{Young tableaux of length } m\}.$$

Conjectures II. Let us suppose that G is a *classical group* i.e. it is of type **A, B, C** or **D.** Note that this is equivalent to supposing that every maximal parabolic subgroup of G is of classical type in the sense defined above. Let l be the rank of G and let us index the maximal parabolic subgroups (containing B) by P_i, $1 \leqslant i \leqslant l$, so that P_i, is associated to the fundamental weight ω_i in the notation of Borel [1]. Suppose that

$$u \in W(G)/W(P_{i_1}),\ v \in W(G)/W(P_{i_2}),\ i_1 \leqslant i_2.$$

Then we define $u \geqslant v$, if

(i) either $i_1 = i_2$; in which case $u \geqslant v$ is just in the sense of the partial order in $W(G)/W(P_{i_1})$, or

(ii) $i_1 \neq i_2$; then if v_1 is the minimal representative in $W(G)$ of v and $\bar{v}_1$ is the canonical image of v_1 in $W(G)/W(P)$, we have

$$u \geqslant \bar{v}_1 \text{ in } W(G)/W(P_{i_1}).$$

Let us denote by $(\tau(i), \varphi(i))$ an *admissible pair* in $W(G)/W(P_i)$ and $P_{(\tau(i), \varphi(i))}$ the element in $H^\circ(G/P_i, L_i)$ associated to $(\tau(i), \varphi(i))$ in conjectures I (L_i being the ample generator of Pic G/P_i). We denote by the same L_i, the line bundle on G/B, which is the pull back of L_i by the canonical morphism $G/B \to G/P_i$). We define $(\tau(i), \varphi(i)) \geqslant (\tau(j), \varphi(j))$ if $i \leqslant j$ and $\varphi(i) \geqslant \tau(j)$, as defined above.

Let $(a) = (a_1, \ldots, a_r)$, $1 \leqslant a_1 < a_2 < \cdots < a_r \leqslant l$, $m = (m_1, \ldots m_r)$, $m_i \geqslant 1$ and $m = \Sigma_{i=1}^r m_i$. We call a *standard monomial of type* $((a), m)$ *(and total degree* $|m|$*)* an

expression of the following type:

$$(*)\quad \left(p_{\tau_1(a_1),\,\varphi_1(a_1)}\, p_{\tau_2(a_1),\,\varphi_2(a_1)} \cdots p_{\tau_{m_1}(a_1),\,\varphi_{m_1}(a_1)}\right) \cdots \left(p_{\tau_1(a_r),\,\varphi_1(a_r)} \cdots p_{\tau_{m_r}(a_r),\,\varphi_{m_r}(a_r)}\right)$$

such that

$$(**)\quad (\tau_1(a_1), \varphi_1(a_1)) \geqslant (\tau_2(a_1), \varphi_2(a_1)) \geqslant \ldots \geqslant (\tau_{m_1}(a_1), \varphi_{m_1}(a_1)) \geqslant (\tau_1(a_2), \varphi_1(a_2)) \ldots$$

The element in (*) can be identified with an element of

$$H^\circ\left(G/B, L_{a_1}^{m_1} \otimes \ldots \otimes L_{a_r}^{m_r}\right)$$

Then the conjecture is that distinct standard monomials as in (*) *form a basis of* $H^\circ\left(G/B, L_{a_1}^{m_1} \otimes \ldots \otimes L_{a_r}^{m_r}\right)$, *when* G *is of type* **A, B** *or* **C.** *Let us call an expression as in* (**) *a Young diagram in* $W(G)$ *of type* $((a), m)$ and total degree $|m|$: The above conjecture implies that

$$\dim H^\circ\left(G/B, L_{a_1}^{m_1} \otimes \ldots \otimes L_{a_r}^{m_r}\right) = \#\,\{\text{Young diagrams in } W(G) \text{ of type } ((a).m)\}$$

We note that an expression as in (**) is precisely a Young diagram in the usual sense when $G=SL(l+1)$ and we identify $W(G)/W(P_j) \approx I_{l+1}(j)$. We observe also that the Conjectures II hold good for type **A**: this is a classical fact when the general field is of characteristic zero (Weyl [23]) further, because of the vanishing theorem it is valid in arbitrary characteristics also.

The Conjectures II can also be stated for G of type **D.** The above definition of standard monomials has to be suitably modified in this case†.

As in I above, a stronger version of these conjectures can be formulated for any Schubert variety in G/B but this has to be done with greater care.

7. Classical invariant theory

We shall now indicate how the results of De Concini and Procesi [5] concerning classical invariant theory can be obtained from the results of section 2 and 5. We shall now work out this method in detail for the case of invariants under the symplectic group. The other cases are treated in a similar way. We shall suppose in the sequel, for simplicity of treatment that char $K \neq 2$ (K = ground field).

Let $X=\text{Spec } R$ be an *affine variety* on which a *reductive algebraic group* G operates. Let $Y=\text{Spec } R^G$ and φ the canonical morphism $\varphi : X \to Y$. One knows, thanks to Haboush's proof of Mumford's conjecture, that we have the following: (Haboush [8]; Seshadri [21])

(i) Y is an *affine variety* (i.e. R^G is a K-algebra of finite type).

†This modification has been done in collaboration with Musili.

(ii) φ is a *good quotient*, in particular, φ is surjective and $\varphi(x_1) = \varphi(x_2)$, $x_i \in X$ if and only if $\overline{0(x_1)}$ and $\overline{0(x_2)}$ intersect, where $\overline{0(x_i)}$ denotes the closure of the G-orbit $0(x_i)$ in X.

(iii) φ is a *categorical quotient* (Mumford [17]; Seshadri [20] and [21].

Let now V denote as in section 2 above, a $2n$ dimensional vector space with basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}$. As in sections 3-5, let G_1 denote the symplectic group $\mathrm{Sp}(2n)$ i.e. the group which leaves the following form invariant:

$$\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} (n \times n \text{ matrix})$$

We denote by $\langle , \rangle$ the skew-symmetric form on V given by the above matrix. Let X denote the affine space

$$X = V \oplus \ldots \oplus + V \ (m \text{ times})$$

and take the diagonal action of G_1 on X. Let $\mathrm{Sk}M_m$ denote the space of skew-symmetric matrices of rank m (as in sections 4 and 5) and let

$$\varphi : X \to \mathrm{Sk}M_m$$

denote the G_1 'invariant' morphism (i.e. for the trivial action of G_1 on SkM_m, φ is a G_1 morphism) defined as follows:

$$\begin{cases} x = (x_1, \ldots, x_m) \in X, \ x_i \in V. \\ \varphi(x) \text{ is the skew symmetric matrix whose } (i, j)\text{th entry is } \langle x_i, x_j \rangle. \end{cases}$$

Then we have the following (parts of which are the known classical theorems of invariants under the symplectic group, when the ground field is of characteristic zero (Weyl [23]).

Theorem 7.1. The image of φ is the determinantal variety D_{2n} (G_2) in $\mathrm{Sk}M_m$ (*cf.* Def. 5.1); further, the canonical map

$$\varphi : X \to D_{2n}(G_2) \text{ (denoted by the same } \varphi)$$

can be identified with the canonical morphism $X \to Y$, $Y = \mathrm{Spec}\ R^G$, $X = \mathrm{Spec}\ R$ described above. In particular, R^G is *generated* by the G-invariant functions $\langle x_k, x_l \rangle$ and by Theorem 5.1, we get a basis of R^G as follows: Let $q_{(i)}$, $(i) = (i_1, \ldots, i_r)$, r *even*, $r \leqslant 2n$, be the Pfaffian of the minor of the skew-symmetric matrix $\varphi(x)$ with rows associated to (i). Then R^G has a basis consisting of the standard monomials in $q_{(i)}$ (*cf.* (iii), Def. 5.1):

$$q_{(i)} q_{(i)'} q_{(i)''} \ldots, (i) \leqslant (i)' \leqslant (i)'' \text{ (note length of (i), (i)', } \ldots, \leqslant 2n).$$

Proof. For this we make use of the following:
Lemma 7.1. Let $Z=\operatorname{Spec} R$ be an affine space on which there is given a *linear action* of a reductive algebraic group G (Seshadri [20] and [21] for the definition of linear actions) and

$$\psi: Z \to \mathbf{A}^N (\mathbf{A}^N = N \text{ dimensional affine space})$$

a *graded G-invariant morphism* (graded means $\psi(tz)=t^r\psi(z)$, $t \in K$). Let $S=\operatorname{Spec} R^G$ and D a *closed subvariety* of $\mathbf{A}^N$ such that $\psi(Z) \subset D$. Then in order that the canonical morphism $\psi : Z \to D$ be identified with the canonical morphism $\pi : Z \to S$, it *suffices* that the following conditions hold:

(i) for $z \in Z^{ss}$ (set of *semi-stable* points in Z, (Seshadri [20] and [21]) $\psi(z) \neq (0)$.

(ii) $\exists$ a non-empty G-stable open subset U of Z such that G operates freely on U, $U \to U$ mod G is a G-principal fibre space and ψ induces an *immersion* of U mod G into D (or $\mathbf{A}^N$).

(iii) D is *normal*

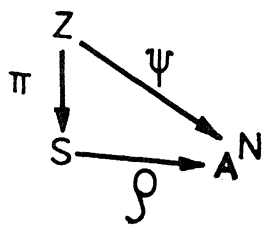
(iv) $\dim U/G = \dim D$.

Proof of lemma. Let $\mathbf{A}^N = \operatorname{Spec} P$. We write

$$Z_1 = \operatorname{Proj} R,\ S_1 = \operatorname{Proj} R^G,\ \mathbf{P}^{N-1} = \operatorname{Proj} P.$$

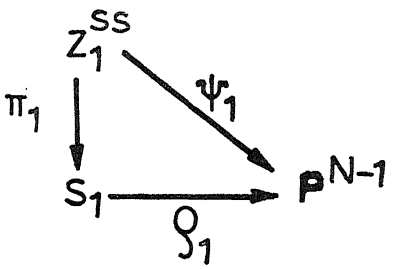
Since ψ is G-invariant, we get a canonical morphism $\rho : S \to \mathbf{A}^N$ such that the following diagram is commutative

(1)



since $\pi : Z \to S$ is a *categorical quotient*. Let Z_1^{ss} denote the open subset of semi-stable points in Z_1. Then we have a canonical morphism $\pi_1 : Z_1^{ss} \to S_1$, which is a categorical quotient (Seshadri [z0] and [21]. Since ψ is graded, ψ defines a rational morphsim $\psi_1 : Z_1 \to \mathbf{P}^{N-1}$ and the hypothesis (i) above implies that ψ_1 is *regular* in Z_1^{ss}. Since $\pi_1 : Z_1^{ss} \to S_1$ is a categorical quotient and ψ_1 is G-invariant, we deduce again a *morphism* $\rho_1 : S_1 \to \mathbf{P}^{N-1}$ and a commutative diagram

(2)



We claim now that ρ_1 is a *finite morphism.* In fact we can find a homogeneous element s in P such that its canonical image $\psi^*(s)$ is a non-zero element of R^G. Now ${\psi_1}^*(s)$ can be identified with a section of an ample line bundle M on S_1 (a power of the tautological line bundle on Z_1^{ss} descends to M). From these considerations it follows easily that the inverse image by ρ_1 of the tautological line bundle on $\mathbf{P}^{N-1}$ is *ample* on S_1. By a familiar argument, this implies that ρ_1 is a finite morphism. The fact that ρ_1 is finite, implies that the morphism ρ itself is finite and this is a consequence of *Lemma* 7.2. Let $\delta : \operatorname{Spec} B \to \operatorname{Spec} K[X_1, \ldots, X_r]$ be a morphism induced by a graded homomorphism (B a graded K-algebra of finite type). Suppose that the rational morphism $\delta_1 : \operatorname{Proj} B \to \operatorname{Proj} K[X_1, \ldots, X_r]$ is a *finite morphism.* Then δ is itself a finite morphism.

Proof of Lemma 7.2. By classical arguments one finds that $\bigoplus_{j \geqslant j_0} B_j$ is a $K[X_1, \ldots, X_r]$ module of finite type for suitable j_0. Since $\bigoplus_{j < j_0} B_j$ is finite dimensional, it follows that B is a $K[X_1, \ldots, X_r]$ module of finite type, which means that δ is finite. This proves Lemma 7.2.

Let us return to the proof of Lemma 7.1. From Lemma 7.2 it follows then that ρ is *finite.* The hypotheses (ii) and (iv) of Lemma 7.1 imply that ρ induces a birational morphism $\rho : S \to D$. Now S is normal and D is normal by (iii). Hence it follows that $\rho : S \to D$ is an isomorphism. This concludes the proof of Lemma 7.1.

We now go back to the proof of Theorem 7.1. We will now check that $\varphi(Z) \subset D_{2n}(G_2)$ and that the conditions of Lemma 7.1 hold (taking $D = D_{2n}(G_2)$, $Z = X$, $\psi = \varphi$ etc.). This is done in the following steps:

1. Let $x^0 = (x_1^0, \ldots, x_m^0) \in X$. Then we claim the following:

$$x^0 \notin X^{ss} \text{(i.e. } x^0 \text{ is } \textit{not} \text{ semi-stable)}$$

$\Longleftrightarrow$ if W is the subspace of V spanned by x_i^0, then W is *totally isotropic* i.e.

$$\langle x_i^0, x_j^0 \rangle = 0 \; \forall \; i, j, \; 1 \leqslant i, j \leqslant m$$

First of all we observe that the above claim implies that the hypothesis (i) of Lemma 7.1 is satisfied for φ.

To prove the above claim, we first note that if W is *not* totally isotropic, then $\langle x_k^0, x_l^0 \rangle \neq 0$ for some k, l with $1 \leqslant k \leqslant m$, $1 \leqslant l \leqslant m$. Let $F_{kl} : X \to K$ be the function

$$F_{kl}(x) = \langle x_k, x_l \rangle, \; x = (x_1, \ldots, x_m).$$

Then F_{kl} is G-invariant and $F_{kl}(x^0) \neq 0$ from which it follows easily that $x^0 \in X^{ss}$. Hence to prove the above claim, it remains to show that

$$W \text{ totally isotropic} \Longrightarrow x^0 \notin X^{ss}.$$

Now the set of maximal isotropic subspaces is a homogeneous space under $G_1 = Sp(2n)$, so that we can find $g \in G_1$ such that $W \subset W_0$ where W_0 is the maximal isotropic subspace of V spanned by the first n coordinates $e_1, \ldots, e_n$. This means that if $gx^0 = (y_1, \ldots, y_m)$, the subspace of V spanned by y_i is contained in W. Since

$$x^0 \in X^{ss} \Longleftrightarrow gx^0 \in X^{ss}$$

it follows, we can suppose, without loss of generality, that $\{x_i^0\}$, $1 \leqslant i \leqslant m$, are in W_0. Let now $\lambda = \lambda(t)$ be the 1-PS (one parameter subgroup) of the maximal torus T of G_1 of the form

$$\begin{pmatrix} t & & & & 0 \\ & \ddots & & & \\ & & t & & \\ & & & t^{-1} & \\ & & & & \ddots \\ 0 & & & & t^{-1} \end{pmatrix}$$

We have

$$x_i^0 = (y_{i1}, \ldots, y_{in}, 0, \ldots, 0), \; y_{ij} \in K$$

and

$$(y_{i1}, \ldots, y_{in}, \ldots 0, , 0) \begin{pmatrix} t & & & & 0 \\ & \ddots & & & \\ & & t & & \\ & & & t^{-1} & \\ & & & & \ddots \\ 0 & & & & t^{-1} \end{pmatrix} = t(y_{i1}, \ldots, y_{in}, 0, \ldots, 0).$$

$$= x^0\lambda.$$

(we take the action of G_1 on X to be on the right).

We deduce that $\lim_{t \to 0} x^0\lambda = (0)$ which implies that $x^0 \notin X^{ss}$. This proves claim I.

II. In order to prove the theorem, we observe first that we can take m to be sufficiently large. In fact, set

$$\text{Spec } R_m = V \oplus \ldots \oplus V(m \text{ times})$$

Then if $m_1 > m_2$, we have canonical G_1-morphisms

$$j : X_{m_1} \to X_{m_2},\ i : X_{m_2} \hookrightarrow X_{m_1}$$

where j is the projection on to the first m_2 factors and i is the inclusion by putting the last m_1–m_2 factors to be zero. If i^*, j^* denote the canonical induced homomorphisms

$$i^* : R_{m_1} \to R_{m_2}, j^* : R_{m_2} \to R_{m_1},$$

we get $i^* \circ j^* =$ identity. Using these maps, it is seen easily that

Theorem 7.1 true for $m_1 \Longrightarrow$ Theorem 7.1 true for m_2.

Hence to prove theorem, *we shall hereafter suppose that* $m \geqslant 2n = \dim V$.

Let U be the open subset of X defined by

$$U = \{x \in X \mid \text{if } x = (x_1, \ldots, x_m),\ x_i \in V, \text{ then the } x_i,\ 1 \leqslant i \leqslant 2n, \text{ are linearly independent in } V\}.$$

Obviously U is G_1-stable and by 1, $U \subset X^{ss}$. We claim now that G_1 operates *freely* on U, $U \to U$ mod G_1 is a principal fibre space and that φ induces an immersion of U mod G_1 into Sk M_m.

To prove this claim, note that we have a G_1-isomorphism

$$U \approx GL(V) \times \underbrace{V \times \ldots \times V}_{m-2n \text{ times}}.$$

From this identification, it follows easily that G_1 operates freely on U. Further one sees that U mod G_1 can be identified with the fibre space, with base ($GL(V)$ mod G_1) and fibre ($V \times \ldots \times V$) ($m-2n$ times), associated to the principal fibre space $GL(V) \to GL(V)$ mod G. Thus to prove the above claim, it suffices to show that φ induces an *immersion* of U mod G_1 into Sk M_n.

Let us first show that φ induces an *injective* map of U mod G_1 into SkM_m. Let $x, y \in U$. Because of the identification $U \approx GL(V) \times (V \times \ldots \times V)$, we can write

$$x = (Y_1, Z_1),\ y = (Y_2, Z_2),\ Y_i \in GL(V).$$

Suppose that $\varphi(x) = \varphi(y)$. If $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$, $x_i, y_j \in V$ this implies, in particular that

$$\langle x_i, x_j \rangle = \langle y_i, y_j \rangle;\ 1 \leqslant i, j \leqslant 2n.$$

This can be written in matrix notation as follows:

$$^tY_1J'Y_1={}^tY_2J'Y_2,\ J'=\begin{pmatrix}0 & J\\ -J & 0\end{pmatrix}.$$

i.e. $^t(Y_1Y_2^{-1})\,J'(Y_1\ Y_2^{-1})=J'$

i.e. $Y_1Y_2^{-1}\in G_1$ which $\Longrightarrow \exists\, g\in G$ such that $Y_1=gY_2$.

Hence to prove that φ induces an injection of U mod G_1 into $\mathrm{Sk}M_m$, we can suppose that

$$x=(Y, Z_1),\ y=(Y, Z_2),\ Y=(x_1, \ldots, x_{2n}).$$

Let $Z_1=(v_1, \ldots)$, $Z_2=(v_2, \ldots)$, $v_i\in V$; Then $\varphi(x)=\varphi(y)$ implies, in particular, that

$$\langle x_i\ v_1\rangle=\langle x_i, v_2\rangle,\ i=1, \ldots, 2n.$$

Since the form $\langle\ ,\ \rangle$ is non-degenerate, it follows that $v_1=v_2$. This implies then that $Z_1=Z_2$ and it follows that φ induces an injection of U mod G_1 into $\mathrm{Sk}M_m$.

We see now easily that the above argument, in fact, proves the stronger fact that if A is *any K-algebra*, φ, induces an *injective map of A-valued points*

$$(U \bmod G_1)\,(A)\to \mathrm{Sk}_m\,(A).$$

This implies that φ induces an immersion of U mod G_1 into $\mathrm{Sk}M_n$ and the above claim in II is proved.

III. It remains to show, since $D_{2n}\,(G_2)$ is normal (*cf.* Theroem 5.1) that

(a) dim U/G=dim $D_{2n}(G_2)$

(b) $\varphi(X)\subset D_{2n}(G_2)$.

By Theorem 5.1 the ideal of $D_{2n}(G_2)$ in $\mathrm{Sk}M_m$ is generated by $q_{(i)}$, $(i)=(i_1, \ldots, i_{2r})$, $2r>2n$. Hence to prove (*b*), it suffices to show that all the $(2r\times 2r)$ minors of $\varphi(x)\in \mathrm{Sk}M_m$, $x\in X$, vanish for $2r>2n$. We leave this as an exercise. The proof of (*a*) is also easy, as the left side of (*a*) is (dim X—dim G) and the right side of (*a*) is also easily calculated by taking the Schubert variety $X(\tau, G_2/Q)$ in $\mathrm{SO}(2n)/Q$ of which $D_{2n}(G_2)$ is the opposite big cell and calculating $l(\tau w_0)$ in $W(G_2)/W(Q)$. We leave this also as an exercise.

This completes the proof of the theorem.

Remark 7.1. A generalisation similar to Theorem 7.1 for the classical theorems on invariants for $GL(n)$ and $\mathbf{O}(2n)$ can be obtained (DeConcini and Procesi [5] Hocheter and Eagon [10]; Kutz [14]) by the same arguments as above and their statements are as follows:

(i) *Case GL(n)*: Let V be the n-dimensional vector space and $G=GL(V)$. Set

$$X=\underbrace{V\oplus\ldots\oplus V}_{m\text{ times}}\oplus\underbrace{V^*\oplus\ldots\oplus V^*}_{m\text{ times}},\ V^*=\text{dual of } V.$$

For $x\in X$, $x=(x_1,\ldots,x_m,\xi_1,\ldots,\xi_m)$ let $\varphi(x)=\|\langle x_i,\xi_j\rangle\|\in M_m$ ($m\times m$ matrices), $\langle\ ,\ \rangle$ being the canonical bilinear form on $V\times V^*$. Let

$$p_{(i),(j)}=\begin{cases}\text{determinant of the minor of } \varphi(x), \text{ corresponding to}\\ (i),(j);(i)=(i_1,\ldots,i_r),\ (j)=(j_1,\ldots,j_r),\ r\leqslant m.\end{cases}$$

Then the morphism φ: $X\to M_m$ is G-invariant, φ maps X on to the *determinantal variety* D_n in M_m (opposite big cell of a Schubert variety in $G_{m,\,2m}$, section 1). Further φ identifies the categorical quotient X mod G with D_n. Applying Theorem 2.1, we get also a basis of R^G, $X=\text{Spec } R$, by *standard monomials* in $p_{(i),(j)}$ with $r\leqslant n$, $(i)=(i_1\ldots,i_r)$ as in that theorem.

(ii) Case $\mathbf{O}(2n)$: (De Concini and Procesi [5]. Let V be a $2n$-dimensional vector space and $\mathbf{O}(2n)$ the orthogonal group leaving $\begin{pmatrix}0 & J\\ J & 0\end{pmatrix}$ invariant. We write $G=\mathbf{O}(2n)$. Set

$$X=V\oplus\ldots\oplus V\ (m\text{ times}),\ X=\text{Spec } R$$

For $x\in X$, $x=(x_1,\ldots,x_m)$, let $\varphi(x)=\|\langle x_i,x_j\rangle\|\in\text{Sym } M_m=$ space of symmetric $(m\times m)$, matrices $\langle\ ,\ \rangle$ being the scalar product on V. Let

$$p_{(i),(j)}=\begin{cases}\text{determinant of the minor of } \varphi(x) \text{ corresponding}\\ \text{to } (i),(j),\ (i)=(i_1,\ldots,i_r),\ (j)=(j_1,\ldots,j_r),\ r\leqslant m.\end{cases}$$

Then the morphism φ: $X\to\text{Sym } M_m$ is G-invariant, φ maps X on to the *determinantal variety* $D_{2n}(G_1)$ in Sym M_m (opposite big cell of a Schubert variety in $\text{Sp}(2n)/Q$, Q maximal parabolic subgroup defined by the right end root, *cf.* Remark 2.1) and φ identifies the categorical quotient X mod G with $D_{2n}(G_1)$. Applying Th. 5.2, we get a basis of R^G by standard monomials in $p_{(i),(j)}$ (*cf.* Def. (ii), 5.1).

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Nonsimilar incompressible laminar boundary layers with magnetic field

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Abstract. The solution of the steady laminar incompressible nonsimilar magnetohydrodynamic boundary layer flow and heat transfer problem with viscous dissipation for electrically conducting fluids over two-dimensional and axisymmetric bodies with pressure gradient and magnetic field has been presented. The partial differential equations governing the flow have been solved numerically using an implicit finite-difference scheme. The computations have been carried out for flow over a cylinder and a sphere. The results indicate that the magnetic field tends to delay or prevent separation. The heat transfer strongly depends on the viscous dissipation parameter. When the dissipation parameter is positive (i.e. when the temperature of the wall is greater than the freestream temperature) and exceeds a certain value, the hot wall ceases to be cooled by the stream of cooler air because the ' heat cushion ' provided by the frictional heat prevents cooling whereas the effect of the magnetic field is to remove the ' heat cushion ' so that the wall continues to be cooled. The results are found to be in good agreement with those of the local similarity and local nonsimilarity methods except near the point of separation, but they are in excellent agreement with those of the difference-differential technique even near the point of separation.

Keywords. Nonsimilar; viscous dissipation; cylinder; sphere; finite difference scheme; skin friction; heat transfer.

1. Introduction

Most laminar boundary layer problems with or without magnetic field are nonsimilar. The nonsimilarity in these cases is due to various factors such as the freestream velocity distribution, transverse curvature etc. The nonsimilar problems are governed by partial differential equations. Since partial differential equations are comparatively difficult to solve, most investigators have solved the problem by employing approximate techniques such as similarity analysis, integral methods, series solutions, local similarity and local nonsimilarity methods (Dewey and Gross 1967; Sparrow *et al* 1970, Sparrow and Yu 1971; Davies 1963; Heiser and Bornhorst 1966). These techniques under suitable approximations allow the governing partial differential equations to be reduced to ordinary differential equations which are then solved numerically or analytically. Various aspects of the steady laminar incompressible boundary layer flow problem with magnetic field for two-dimensional bodies with or without pressure gradient have been considered by several authors (Davies 1963; Heiser and Bornhorst 1966; Rossow 1957; Cess 1960; Sherman 1961; Glauert 1961; Hildyard 1972) using one of the above-mentioned mathematical techniques. To the authors' knowledge, the solutions of the magnetohydrodynamic problems with viscous dissipation over two-dimensional and axisymmetric bodies using finite-difference schemes have not been reported in the literature.

In this paper, we have studied the nonsimilar laminar steady incompressible two-dimensional and axisymmetric magnetohydrodynamic boundary-layer flow and heat transfer problem with viscous dissipation for electrically conducting fluids with magnetic field and pressure gradient. The partial differential equations governing the flow were first transformed to a co-ordinate system having a finite range and then solved by using an implicit finite-difference scheme (Marvin and Sheaffer 1969; Vimala and Nath 1975). Computations were carried out for two problems, namely, cylinder in cross flow and the flow over a sphere. The results have been compared with those of local similarity, local nonsimilarity, and difference-differential methods (Sparrow *et al* 1970; Sparrow and Yu 1971; Terrill 1960; Smith and Clutter 1963).

2. Governing equations

We consider steady laminar incompressible electrically conducting fluid flow past a two-dimensional or an axisymmetric surface with an applied magnetic field. It is assumed that the fluid has constant physical properties. The magnetic Reynolds number is assumed to be small. Hence, the induced magnetic field is negligible in comparison to the applied magnetic field. We also assume that there is no external applied electric field. The Hall current and displacement current effects have also been neglected. For these conditions, the boundary layer equations with constant freestream temperature in dimensionless form governing the nonsimilar flow taking into account the effects of the viscous dissipation and Joule heating can be expressed as (Rossow 1957; Cess 1960).

$$f''' + ff'' + \beta(\xi)(1-f'^2) + \beta_1(\xi)M(1-f') = 2\xi(f' f_\xi' - f'' f_\xi) \tag{1}$$

$$g'' + Pr\, f g' - Br\,(U/U_\infty)^2[f''^2 + \beta_1(\xi)M(f'^2-1)]$$

$$= 2\,Pr\,\xi(f'\, g_\xi - g' f_\xi) \tag{2}$$

with boundary conditions

$$f(\xi, 0) = f'(\xi, 0) = g(\xi, 0) = 0, f'(\xi, \infty) = g(\xi, \infty) = 1 \tag{3}$$

where

$$\xi = \int_0^x [U(x)/U_\infty)(r/L)^{2j} d(x/L),\ \eta = y(r/L)^j U(x)/(2\nu L U_\infty \xi)^{1/2} \tag{4a}$$

$$\psi(x, y) = (2\nu L U_\infty \xi)^{1/2} f(\xi, \eta),\ u = (L/r)^j \psi_y,\ v = -(L/r)^j \psi_x \tag{4b}$$

$$g = (T-T_w)/(T_\infty - T_w),\ M = Hm^2/Re = \sigma B_0^2 L/(\rho U_\infty) \tag{4c}$$

$$\beta(\xi) = 2(L/r)^{2j}[(dU/dx)/U^2] \int_0^x U(r/L)^{2j}\, dx \tag{4d}$$

$$\beta_1(\xi) = 2\xi(U_\infty/U)^2(L/r)^{2j},\ Br = \mu U_\infty^2/[K(T_w - T_\infty)]. \tag{4e}$$

Here ξ, η are the transformed co-ordinates; ψ and f are the dimensional and dimensionless stream functions, respectively; g is the dimensionless temperature; β is the pressure gradient parameter; β_1 is a function depending on the streamwise distance ξ; M is magnetic parameter; Hm is the Hartmann number, $Re(=U_\infty L/\nu)$ is the Reynold number; Br is the dissipation parameter (Brinkman number); T is the temperature; B_0 is the applied magnetic field along y direction; σ is the electrical conductivity; ρ is the density; L is the characteristic length; x and y are distances along and perpendicular to the surface, respectively; u and v are the velocity components along x and y directions, respectively; μ and ν are the coefficient of viscosity and kinematic viscosity respectively; K is the thermal conductivity; r is the radius of revolution of an axisymmetric body; U is the potential velocity at the edge of the boundary layer; Pr is the Prandtl number; $j=0$ for a two-dimensional body and $j=1$ for an axisymmetric body; the subscripts w and ∞ denote wall and freestream values, respectively; the subscripts x, y and ξ denote derivatives with respect to x, y and ξ, respectively and prime denotes differentiation with respect to η. It may be noted that eqs (1) and (2) reduce to classical boundary-layer equations when $M=0$.

The dimensionless skin-friction and heat-transfer coefficients are given by (Sparrow 1970).

$$C_f(\bar{Re})^{1/2}=2(r/L)^j\,[xU/(2\xi LU_\infty)]^{1/2}\,f''_w \tag{5a}$$

$$Nu(\bar{Re})^{-1/2}=(r/L)^j\,[xU/(2\xi LU_\infty)]^{1/2}\,g'_w \tag{5b}$$

with

$$C_f=2\tau_w/\rho U^2,\ Nu=x(\partial T/\partial y)_w/(T_\infty-T_w),\ \bar{Re}=Ux/\nu \tag{5c}$$

where C_f is the skin-friction coefficient, Nu is the Nusselt number, τ_w is the shear stress at the wall and $\bar{Re}$ is the local Reynolds number.

3. Transformation to finite co-ordinates

We transform eqs (1) and (2) to a new system of co-ordinates wherein the infinite range of integration (0, ∞) on η is replaced by a finite range (0, 1) on z which is more convenient from the viewpoint of computation. The transformation is given by (Marvin and Sheaffer 1969).

$$z=1-\exp(-\alpha\eta) \tag{6}$$

where the scaling factor α provides an optimum distribution of nodal points across the boundary layer. We define $f'=F$ and $\alpha(1-z)=\bar{\eta}$ and change the variable η to z. Consequently, eqs (1) and (2) reduce to

$$\bar{\eta}^2F_{zz}+\bar{\eta}(f-\alpha)F_z+\beta(\xi)(1-F^2)+\beta_1(\xi)M(1-F)$$

$$=2\xi(FF_\xi-\bar{\eta}f_\xi F_z) \tag{7}$$

$$\bar{\eta}^2 g_{zz}+\bar{\eta}(Pr\, f-\alpha)g_z-Br(U/U_\infty)^2\,[\bar{\eta}^2 F_z^2+\beta_1(\xi)M(F^2-1)]$$

$$=2Pr\;\xi\,(F\,g_\xi-\bar{\eta}\,f_\xi\,g_z). \tag{8}$$

The boundary conditions (3) can now be written as

$$F(\xi, 0)=g(\xi, 0)=0;\; F(\xi, 1)=g(\xi, 1)=1. \tag{9}$$

The solution of the partial differential eqs (7) and (8) under conditions (9), which govern the nonsimilar flow past an arbitrary two-dimensional or axisymmetric body can be obtained by using an implicit finite-difference scheme provided (U/U_∞), $\beta(\xi)$ and $\beta_1(\xi)$ (which depend on the shape of the body) are prescribed. Since the method is described in full detail in Marvin and Sheaffer (1969) and Vimala and Nath (1975), for the sake of brevity, it is not repeated here. In particular, we have solved the above equations for the case of a cylinder and a sphere.

4. Cylinder in cross flow

We consider the flow of an electrically conducting fluid over a circular cylinder with an applied magnetic field where the nonsimilarity arises due to the external velocity distribution at the edge of the boundary layer. This velocity is given by the potential theory

$$U/U_\infty=2\sin\bar{x},\; \bar{x}=x/R \tag{10a}$$

where the pressure is determined by

$$-\rho^{-1}(\partial p/\partial x)=U(dU/dx)+\sigma B_0^2(u-U)/\rho. \tag{10b}$$

Here p is the pressure; R and $\bar{x}$ are the radius of the cylinder and dimensionless distance, respectively. We now take R as the reference length instead of L and put $j=0$. Consequently, eqs (4a, d, e) and (5) using (10a) reduce to (Sparrow *et al* 1970)

$$\xi=2(1-\cos\bar{x}),\; \beta(\bar{x})=2\cos\bar{x}/(1+\cos\bar{x}),\; \beta_1(\bar{x})=1-\beta(\bar{x})/2 \tag{11a}$$

$$C_f(\bar{Re})^{1/2}=[2\,\bar{x}\,\sin\bar{x}/(1-\cos\bar{x})]^{1/2}\,f''_w \tag{11b}$$

$$\mathrm{Nu}(\bar{Re})^{-1/2}=[(\bar{x}/2)\sin\bar{x}/(1-\cos\bar{x})]^{1/2}\,g'_w. \tag{11c}$$

Equations (7) and (8) with (10a) and (11) under conditions (9) were solved for various values of M and Br using the implicit finite-difference scheme mentioned earlier. The step sizes $\triangle z=0{\cdot}005$ and $\triangle\bar{x}=0{\cdot}05$ were used for computational purposes and further reduction in them produced changes only in the fourth significant place. But in the neighbourhood of the point of separation $\triangle z=0{\cdot}0001$ and $\triangle\bar{x}=0{\cdot}01$ were used. The scaling factor α was taken as $0{\cdot}5$.

The velocity and temperature profiles are shown in figures 1 and 2. The velocity

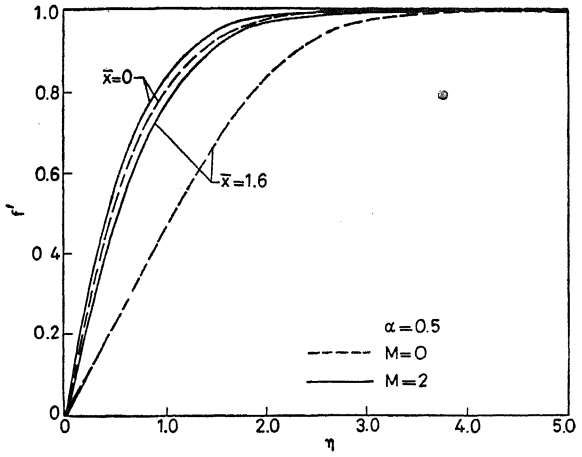

Figure 1. Velocity profiles (cylinder).

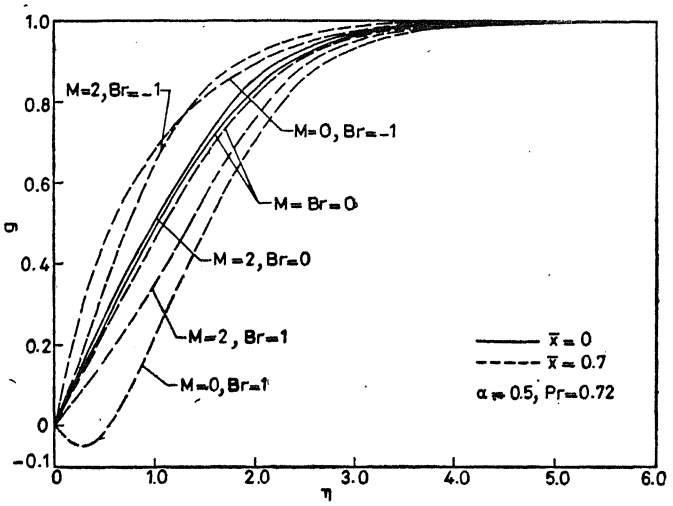

Figure 2. Temperature profiles (cylinder).

profiles become less steep as $\bar{x}$ increases, but the reverse is true when the magnetic parameter M increases. In the presence of the magnetic field, the profiles are comparatively less sensitive to the change in $\bar{x}$. It may be noted that the velocity profiles are independent of the dissipation parameter Br, but the temperature profiles strongly depend on Br. It may be remarked that in eq. (8), Br occurs in combination with U/U_∞ and $U/U_\infty=0$ at $\bar{x}=0$. Hence, eq. (8) at $\bar{x}=0$ becomes independent of Br whether M is zero or non-zero. Therefore for a given M, the temperature profiles for $Br=1$ or -1 at $\bar{x}=0$ are the same as those of $Br=0$. Further, when $Br=0$, the temperature profiles change little due to the change in $\bar{x}$ except near the point of separation. In fact, when the magnetic field is imposed, the change is still smaller and hence the temperature profiles for $M=2$ at $\bar{x}=0{\cdot}7$ could not be shown in figure 2. It is evident from figure 2 that for $M=0$ and $Br=1$ ($T_w>T_\infty$), the temperature $g(g=(T-T_w)/(T_\infty-T_w))$ for $\bar{x}\geqslant 0{\cdot}5$ (approximately) becomes negative near the wall which implies that the fluid near the wall is warmer than the wall itself owing to the

generation of frictional heat. In such cases, the wall will not be cooled by the strean of air flowing past it. Similar effects have also been observed for the case of a fla plate without magnetic field (Schlichting 1968). It is also observed that as Br increases the point at which $g<0$ moves upstream. On the other hand, the effect of th magnetic field is to prevent the generation of frictional heat and the fluid near th wall remains cooler than the wall and the wall continues to be further cooled. Thus the magnetic field and the dissipation parameter play opposite roles as far as heatin or cooling effect on the wall is concerned.

The variation of the skin-friction parameter f''_w with $\bar{x}$ is given in figure 3. Lik

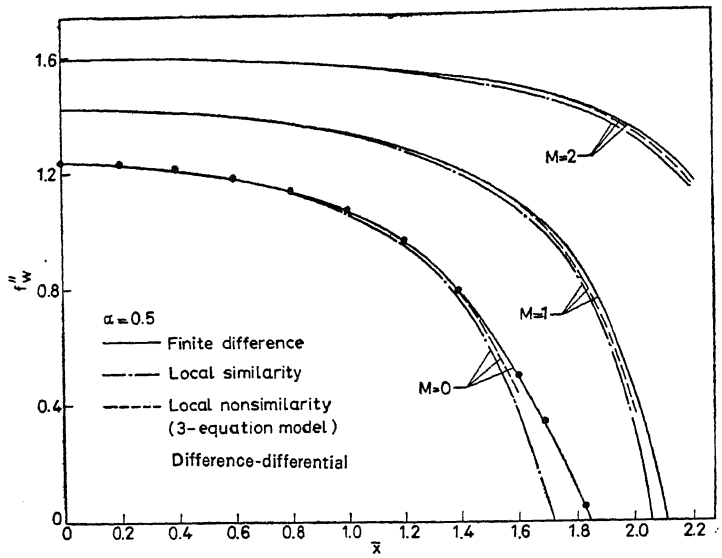

Figure 3. Variation of skin-friction with $\bar{x}$ (cylinder).

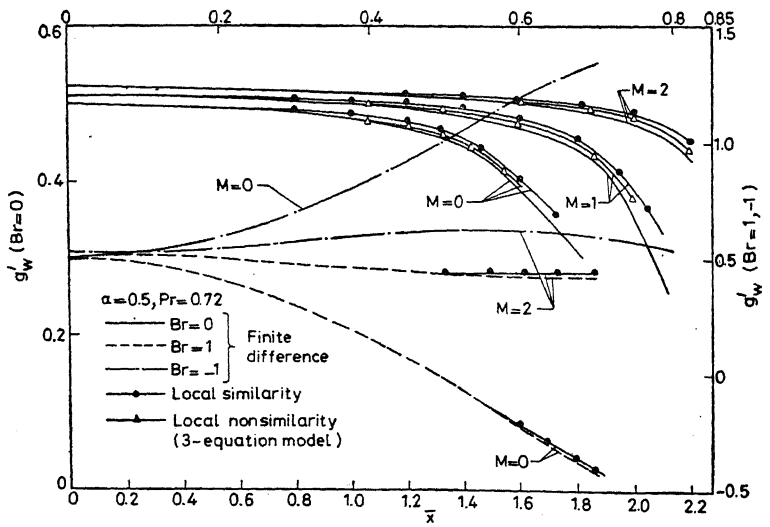

Figure 4. Variation of heat-transfer with $\bar{x}$ (cylinder).

velocity profiles, f''_w is also independent of Br. As expected f''_w decreases as $\bar{x}$ increases whatever may be the values of M. At a given $\bar{x}$, f_w'' increases as M increases. It is clearly seen from figure 3 that the effect of the magnetic field is to delay or prevent separation. The variation of the heat-transfer parameter g'_w with $\bar{x}$ is depicted in figure 4. For the sake of convenience, the results for $Br=0$ and $Br\neq 0$ are shown on a different scale. When $Br=0$, g'_w increases as M increases. However, as $\bar{x}$ increases, it remains nearly constant in the beginning and then decreases. When $Br=1$ and $M=0$, g'_w decreases as $\bar{x}$ increases and becomes negative at a certain $\bar{x}$ (the physical significance of g'_w becoming negative has been explained earlier). On the other hand when $Br=-1$ and $M=0$, g'_w increases as $\bar{x}$ increases. In the presence of the magnetic field, the change in g'_w with $\bar{x}$ is comparatively small. To check the accuracy of the present finite-difference method, we have compared our skin-friction results for $M=0$ with those of Terrill (1960) who used the difference-differential scheme and found them in excellent agreement (see figure 3). We have also compared our results with those of the local similarity and local nonsimilarity solutions and found them in good agreement except near the point of separation where these methods are not expected to give accurate results (Sparrow *et al* 1970). As expected, the local nonsimilarity solutions have been found to be more accurate than the local similarity solutions. It is further observed that the local similarity and local nonsimilarity results underestimate the skin friction but overestimate the heat transfer.

5. Flow over a sphere

We now consider the flow of an electrically conducting fluid over a sphere with an applied magnetic field, where the non-similarity is due both to external freestream velocity and to curvature of the body. In this case, the velocity at the edge of the boundary layer is given by

$$U(x)/U_\infty = 1{\cdot}5 \sin \bar{x},\ r/R = \sin \bar{x} \tag{12}$$

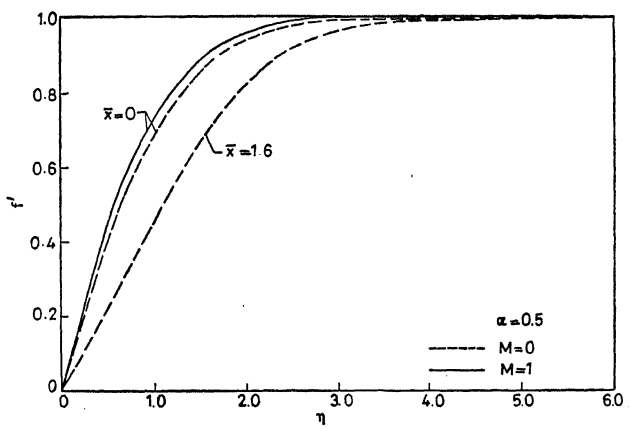

Figure 5. Velocity profiles (sphere).

and the pressure gradient is governed by (10b). Like for the cylinder, eqs (4a, d, e) and (5) taking $j = 1$ reduce to

$$\xi = (2-3\cos\bar{x}+\cos^3\bar{x})/2, \quad \beta(\bar{x})=2[\cos\bar{x}\,(\cos\bar{x}+2)/(1+\cos\bar{x})^2]/3 \quad (13a)$$

$$\beta_1(\bar{x}) = 4\,[(\cos\bar{x}+2)/(1+\cos\bar{x})^2]/9,$$

$$C_f\,(Re_x)^{1/2} = 2\beta_2(\bar{x})\,f''_w, \quad Nu\,(Re_x)^{-1/2} = \beta_2(\bar{x})\,g'_w \quad (13b)$$

$$\beta_2(\bar{x}) = \sin\bar{x}\,[(3\,\bar{x}\,\sin\bar{x})/\{2(2-\cos\bar{x}+\cos^3\bar{x})\}]^{1/2} \quad (13c)$$

The velocity and temperature profiles are shown in figures 5–6 and the skin-friction and heat-transfer parameters in figures 7 and 8. It may be remarked that qualitatively the results are similar to those of the cylinder, hence no detailed discussion of the

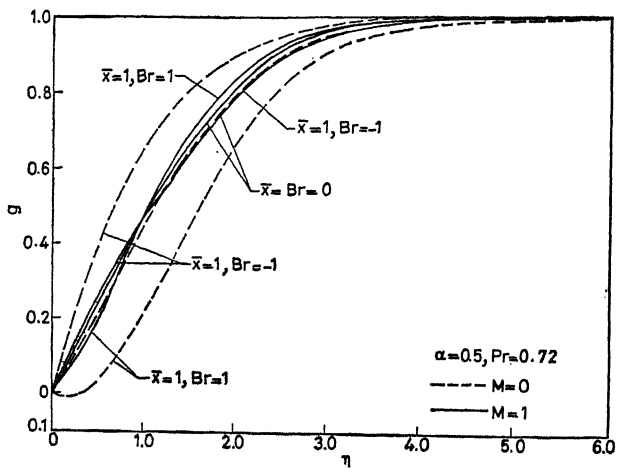

Figure 6. Temperature profiles (sphere).

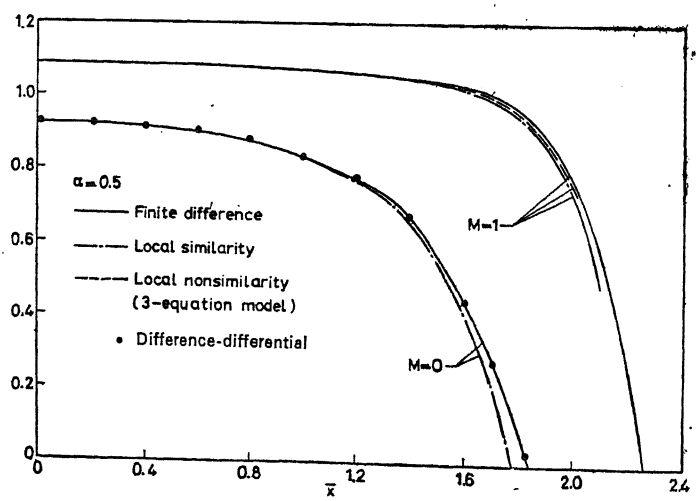

Figure 7. Variation of skin friction with $\bar{x}$ (sphere).

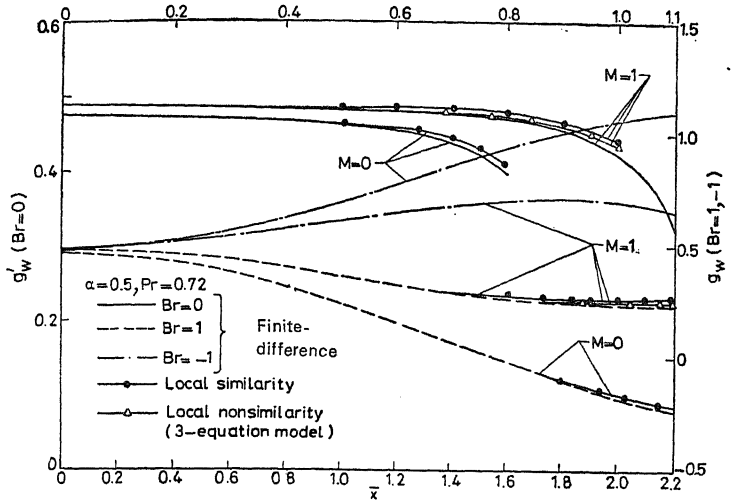

Figure 8. Variation of heat transfer with $\bar{x}$ (sphere).

results is given for this case. Our skin-friction results for $M=0$ are found to be almost identical with those of Smith and Clutter (1963) who used the difference-differential technique provided we divide their results by $2^{1/2}$ (see figure 7). This factor is due to slightly different sets of transformations used by Smith and Clutter (1963). Like cylinder, the results are found to be in good agreement with those of the local similarity and local nonsimilarity methods except near the point of separation.

6. Conclusions

The dissipation parameter for the case when the wall temperature is greater than the freestream temperature in the absence of a magnetic field generates frictional heat which makes the fluid near the wall warmer than the wall itself and thus prevents the wall being cooled. On the other hand, the magnetic field prevents the generation of frictional heat so that the fluid near the wall remains cooler than the wall. Further, the magnetic field tends to delay or prevent separation. The results are found to be in good agreement with those of the local similarity and local nonsimilarity methods except near the point of separation, but they are in excellent agreement with those of the difference-differential technique even near the point of separation.

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Composite expansion solution of fingero-imbibition in double phase flow through underground porous medium

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Abstract. The phenomenon of fingero imbibition due to simultaneous occurrence of imbibition and instabilities in double phase flow through homogeneous porous media is discussed by the method of composite expansion. The injected liquid is less viscous and preferentially wetting with respect to the porous media, and coefficient of saturation is considered as constant.

Keywords. Composite expansion; fingero imbibition; porous media.

1. Introduction

It is well known that under certain circumstances the phenomena of instabilities (fingering) and imbibition (Scheidegger 1960a) may occur simultaneously in a displacement process involving two immiscible fluids. This is referred to as fingero imbibition (Scheidegger 1961). The phenomena of fingering and imbibition, whether occurring singly or simultaneously in displacement processes, have gained much importance particularly in petroleum technology and hydrogeology, and many authors have investigated them from different aspects; for example, Graham and Richardson (1959), Scheidegger (1960b, 1969), Rijik (1960), Verma (1968, 1969a,b, 1970).

In this paper the underlying assumptions are that the two fluids are immiscible, and the injected liquid (I) is less viscous as well as preferentially wetting with respect to the porous medium. The mathematical formulation of the basic equation yields a nonlinear differential equation whose solution has been obtained by the method of composite expansion for the case of constant saturation coefficient.

2. Formulation and solution

We consider that a cylindrical piece of homogeneous porous matrix, saturated with native liquid N is completely surrounded by an impermeable surface except for one end of the cylinder which is designated as the imbibition face ($x=0$) and this end is exposed to an adjacent formulation of injected liquid I. If the injected liquid I is less viscous and in preferentially wetting phase, the phenomenon of fingering will occur simultaneously with imbibition, and then this arrangement describes a one-dimensional fingero-imbibition phenomenon in which the injection is started by imbibition and the resulting displacement produces instabilities (Verma 1970b).

Assuming that Darcy's law is valid in the case being investigated, we may write the basic flow equations governing imbibition phenomenon as

$$v_i = -\frac{k_i}{\delta_i} K \frac{\partial p_i}{\partial x} \tag{1}$$

$$v_n = -\frac{k_n}{\delta_n} K \frac{\partial p_n}{\partial n} \tag{2}$$

$$v_i = -v_n \tag{3}$$

$$P_c = P_n - P_i \tag{4}$$

$$P \frac{\partial S_i}{\partial t} + \frac{\partial v_i}{\partial x} = 0 \tag{5}$$

where v_i and v_n are velocities, k_i and k_n are relative permeabilities, δ_i and δ_n are the viscosities, P_i and P_n are the phase pressures of the injected and native liquid, respectively; P and K are the porosity and permeability of the medium; S_i is the saturation of injected liquid, P_c is capillary pressure, x is linear coordinate, and t is the time. Combining eqs (1)–(5), we get

$$P \frac{\partial S_i}{\partial t} + K \frac{\partial}{\partial x}\left[\frac{k_i k_n}{\delta_i k_n + \delta_n k_i} \frac{dp_c}{dS_i} \frac{\partial S_i}{\partial x}\right] = 0. \tag{6}$$

It is well known from the literature (Scheidegger 1961) that the value of P_c may be written as

$$P_c = \beta g(S_i) \tag{7}$$

where β is the constant capillary pressure coefficient which is regarded as a small (Verma 1969a) a parameter in the present investigation. Substituting this value into eq. (6), we get,

$$P \frac{\partial S_i}{\partial t} - \beta K \frac{\partial}{\partial x}\left[D(S_i) \frac{\partial S_i}{\partial x}\right] = 0 \tag{8}$$

where $D(S_i) = [k_i k_n / \delta_i k_n + \delta_n k_i]\, g'(S_i)$, because k_i and k_n are functions of S_i only (Scheidegger 1960a), and the negative sign of the second term is due to the fact that P_c is a decreasing function of S_i (Muskat 1949).

A set of boundary conditions may be chosen as

$$S_i(0, t) = S_{io}(\beta t),\ t > 0 \tag{9}$$

$$S_i(x, 0) = \phi(x)\ 0 < x \leqslant L \tag{10}$$

where the first boundary condition defines the imbibition face with saturation S_{io} and the second gives the initial saturation of injected liquid in the porous matrix.

It may be remarked that, in general, the porous matrix may have some initial saturation for the displacing liquid; for example, presence of connate water in an oil reservoir (Muskat 1949).

Using the dimensionless variable $z=x/L$ and $\theta=\beta K/PL^2$ eq. (8), we get

$$\frac{\partial S_i}{\partial \theta}=\frac{\partial}{\partial z}\left[\ D\ (S_i)\ \frac{\partial S_i}{\partial z}\right]. \tag{11}$$

This is a nonlinear differential equation for the saturation; in addition one has the boundary conditions:

$$S_i\ (0,\ \theta) = S_{io}\ (\beta\theta)\ \theta > 0,$$

$$S_i\ (z,\ 0) = \phi\ (z)\ 0 < z \leqslant 1.$$

For a further discussion of the problem, since $D(S_i)$ is function of saturations and $0 \leqslant S_i \leqslant 1$, we can consider an average value of $D(S_i)$ constant (Verma 1970b). We assume here that the average value of $D(S_i) \equiv \overline{D}(S_i)$ is a constant and that the saturation $S_i\ (z,\ \theta,\ \beta)$ depends on one space variable (i.e. z) which ranges from 0 to $b(\beta\theta)$, where $b(\beta\theta)$ is a slowly varying function of θ. Then the mathematical description of the probem is

$$\frac{\partial S_i}{\partial \theta} = \overline{D}\frac{\partial^2 S_i}{dz^2},\ 0 \leqslant z \leqslant b(\beta\theta) < 1 \tag{12}$$

$$S_i\ (0,\ \theta) = S_{io}\ (\beta\theta),\ S_i\ [b(\beta\theta),\ \theta] = 0 \tag{13}$$

$$S_i\ (z,\ 0) = \phi(z),\ 0 \leqslant z \leqslant b(0). \tag{14}$$

If one changes from the variable θ to $\tau=\beta\theta$, eqs (12) and (13) become

$$\beta S_i,\ \theta = \overline{D} S_i,\ zz \quad 0 \leqslant z \leqslant b(\tau) < 1 \tag{15}$$

$$S_i\ (0,\ \tau) = S_{io}(\tau)\ S_i\ [b(\tau),\ \tau] = 0 \tag{16}$$

where $S_i,\ \theta = \dfrac{\partial S_i}{\partial \theta}$ and $S_i,\ zz = \dfrac{\partial^2 S_i}{dz^2}$.

Since β multiplies $\partial S_i/\partial\theta$, the straightforward perturbation solution for small (Mishra and Verma 1974), β, keeping τ fixed, cannot satisfy in general the initial condition (14) and uniformly exists at the end near $\tau=0$. To describe the behaviour of S_i near $t=0$, a stretching transformation $\theta=\tau/\beta$ is needed. As verified below, the special function $\exp\ [-g\ (\tau)/\beta]$ with $g(\tau)\to\tau$ as $\tau\to 0$ describes the behaviour of S_i in his region. Therefore, we assume that S_i has a uniformly valid asymptotic expansion of the form

$$S_i = \sum_{n=0}^{\infty} \beta^n f_n\ (z,\ \tau) + \exp\ [-g(\tau)/\beta] \sum_{n=0}^{\infty} h_n\ (z,\ \tau). \tag{17}$$

Substituting the expansion into (14) through (16) and equating to zeroth coefficients of β^n and $\beta^n \exp[-g(\tau)/\beta]$ for all n, we obtain

$$f_0, zz=0, f_0(0, \tau) = S_{i0}(\tau), f_0[b(\tau), \tau] = 0 \tag{18}$$

$$\bar{D}h_{0,zz}+g'n_0 = 0, h_0(0, \tau) = 0, h_0[b(\tau), \tau] = 0 \tag{19}$$

$$f_0(z, 0)+h_0(z, 0)=\phi(z)\ 0 \leqslant z \leqslant b(0) \tag{20}$$

and for $n \geqslant 1$

$$\bar{D}f_{n,zz}= f_{n-1}, f_n(0, \tau)= f_n[b(\tau), \tau] = 0 \tag{21}$$

$$\bar{D}h_{n,zz}+g' h_n= h_{n-1,\tau}\ h_n(0, \tau)= h_n[b(\tau), \tau] = 0 \tag{22}$$

$$f_n(z, 0)+h_n(z, 0) = 0 \tag{23}$$

where g' stands for $dg/d\tau$.
The solution of eq. (18) is

$$f_0= s_{i0}(\tau)\left[1-\frac{z}{b(\tau)}\right]. \tag{24}$$

Since the boundary conditions in h_o are homogeneous, the equation for h_o a non-trivial solution if and only if g' is one of the eigenvalues

$$g'_k=\left[\frac{k\pi\sqrt{\bar{D}}}{b(\tau)}\right]^2, k=1, 2, 3, \ldots. \tag{25}$$

Then the corresponding normalized eigenfunctions are

$$k=\left[\frac{2}{b(\tau)}\right]^2 \sin\frac{k\sqrt{\bar{D}}.\pi x}{b(\tau)} \tag{26}$$

Hence $h_o=\bar{D}a_o(\tau)\chi_k(z, \tau)$ (27)

where a_o is an unknown function which is determined from the examination of equation for n_1

with h_o known, eq. (22) becomes for $n=1$

$$\bar{D}h_{1,zz}+g'_k h_1=\bar{D}(a'\chi_k+a_o\chi_{k,\tau}) \tag{28}$$

$$h_1(0, \tau)=h_1[b(\tau), \tau] =0.$$

We assume that h_1 can be expanded in terms of the eigenfunction χ_k; that is

$$h_1=\bar{D}\sum_{m=1}^{\infty} C_m(\tau)\chi_m(z, \tau). \tag{29}$$

Substituting eq. (29) into eq. (28) and using the fact that

$X_{m,\,zz}=-g'_m X_m$, we obtain

$$\sum_{m=1}^{\infty}(g'_k-g'_m)C_m X_m=a'_o X_k+a_o X_{k,\,\tau}. \tag{30}$$

If we multiply this equation by X_k and integrate from $z=0$ to $b(\tau)$, the left-hand side vanishes because X_k is orthogonal to X_n for $k\neq m$, and $g'_m=g'_k$ where $k=m$, therefore

$$\int_0^{b(\tau)}(a'_0 X_k^2+a_0 X_{k,\tau}X_k)\,dz=0 \tag{31}$$

which is the condition for solving (28). Since

$$\int_0^{b(\tau)} X_k^2\,dz=1 \tag{32}$$

$$\frac{d}{d\tau}\int_0^{b(\tau)} X_k^2\,dz=0\;b'(\tau)\,X^2{}_k\,[b(\tau),\,\tau]+2\int_0^{b(\tau)} X_k\,X_{k,\tau}\,dz=0. \tag{33}$$

Since $X_k\,[b\,(\tau),\tau]=0$

$$\int_0^{b(\tau)} X_k\,X_{k,\tau}\,dz=0 \tag{34}$$

Hence (31) and (32) lead to

$a_o=a$ constant.

Therefore, the zeroth order solution is

$$S_i\,(z,\,\theta;\,\beta)=S_{io}\,(\tau)\left[1-\frac{z}{b(\tau)}\right]+\bar{D}\sum_{k=1}^{\infty}a_k\left[\frac{2}{b(\tau)}\right]^{1/2}$$

$$\sin\frac{k\sqrt{\bar{D}}\cdot\pi z}{b\,(\tau)}\;\exp\left[-\frac{\bar{D}\,k^2\pi^2}{\beta}\int_0^{\tau}b^{-2}\,(v)\,dv\right]+0(\beta) \tag{35}$$

where a_k is constant given by

$$a_k=\left[\frac{2}{b\,(0)}\right]^{1/2}\int_0^{b\,(0)}\left[\phi\,(z)-S_{io}\,(0)\left(1-\frac{z}{b\,(0)}\right)\right]$$

$$\sin\left[\sqrt{\bar{D}}\cdot k\pi z/b(0)\right]\cdot dz \tag{36}$$

3. Concluding remarks

In this paper we have obtained a composite expansion for the fingero imbibition phenomenon which arises due to the lesser viscosity and preferential wettability of the

displacing fluid in a two-phase flow, e.g. oil and water, through porous media. In particular, we have obtained the saturation of the wetting phase when the saturation coefficient is assumed to be constant, and this represents the average cross sectional area occupied by the fingers (Verman 1970b). We have not included here the numerical and graphical illustration due to our particular interest.

Notwithstanding the analytical interest of the authors, it is believed that the paper will be relevant to atleast certain practical problems. e.g. petroleum technology.

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Unsteady hydromagnetic motion of a rotating conducting fluid

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Abstract. The combined effects of stratification and magnetic field on the unsteady motion of a viscous, electrically conducting fluid between two rotating disks are analysed. Solutions are obtained for the linearized equations under Boussinesq approximation and steady state solutions are deduced from them. The results are compared with those obtained by Loper and Benton and Balan *et al.* Graphs are presented for the steady state velocity, magnetic field and temperature distributions.

Keywords. Unsteady motion; hydromagnetic motion; stratified flow; rotating flow; motion.

1. Introduction

The study of rotating fluids has gained considerable interest and importance owing to its application in geophysical phenomena. The effect of stratification on steadily rotating fluid has been studied by Carrier (1965) and Barcilon and Pedlosky (1967) and on unsteady motion by Allen (1971), Walin (1969), Hendershott (1969). The problem was extended by Niimi (1971) and Balan *et al* (1973) taking into account temperature variation along the boundaries. The influence of axial magnetic field on unsteady rotating homogeneous electrically conducting fluid between two horizontal disks has been investigated by Loper and Benton (1969).

In this paper we analyse the combined effects of stratification, a uniform axial magnetic field and temperature variation along the boundaries, on the unsteady motion of a rotating electrically conducting fluid between two horizontal disks which are maintained at different temperatures. Following the method of earlier investigators, the governing equations are linearized under Boussinesq approximation and Laplace Transform technique is employed.

2. Formulation of the problem

We consider the motion of an electrically conducting viscous incompressible fluid between two infinite parallel disks which are rotating about a common vertical axis with slightly different angular velocities Ω_1 and Ω_2 and are maintained at temperatures T_1' and T_2' respectively. The distance of separation between the disks is denoted by $2L$. We choose cylindrical polar coordinates (r', θ', z') with z' axis along the axis of rotation and origin midway between the disks.

The linearized equations governing the axially symmetric motion induced by the

small rotation in the presence of a transverse magnetic field H_0 with respect to a coordinate frame rotating with angular velocity Ω $[=(\Omega_1+\Omega_2)/2]$ in nondimensional form are:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}-2v=-\frac{\partial p}{\partial r}+E\mathscr{L}u+S^2\left(\frac{\partial h_r}{\partial z}-\frac{\partial h_z}{\partial r}\right) \tag{2}$$

$$\frac{\partial v}{\partial t}+2u=E\mathscr{L}v+S^2\frac{\partial h_\theta}{\partial z} \tag{3}$$

$$\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+E\nabla^2 w+T \tag{4}$$

$$f_R\frac{\partial T}{\partial t}+w=\frac{E}{\sigma}f_R\nabla^2 T \tag{5}$$

$$\frac{\partial h_r}{\partial t}=\frac{\partial u}{\partial z}+E_m\mathscr{L}h_r \tag{6}$$

$$\frac{\partial h_\theta}{\partial t}=\frac{\partial v}{\partial z}+E_m\mathscr{L}h_\theta \tag{7}$$

$$\frac{\partial h_z}{\partial t}=-\frac{1}{r}\frac{\partial}{\partial r}.(ru)+E_m\nabla^2 h_z \tag{8}$$

where

$$\nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2},\quad \mathscr{L}\equiv\nabla^2-\frac{1}{r^2}$$

$$E=\frac{\nu}{\Omega L^2}\ \text{(Ekman number)}$$

$$f_R=\frac{\Omega^2}{g\beta\alpha^2}\ \text{(Internal Froude number)}$$

$$\sigma=\frac{\nu}{\kappa}\ \text{(Prandtl number)}$$

$$S^2=\frac{\mu_e H_0^2}{\rho_0\Omega L^2}\ \text{(Magnetic interaction parameter)}$$

$$E_m=\left(\frac{1}{\mu_e\sigma_e\Omega L^2}\right),\ \text{magnetic Ekman number}$$

ν = Kinematic coefficient of viscosity

κ = Thermal diffusivity coefficient

μ_e, σ_e=Permeability and conductivity of the fluid

α = Coefficient of thermal expansion

g = acceleration due to gravity, $\beta = \dfrac{(T_1'-T_2')}{2L}$.

The physical quantities have been rendered nondimensional as follows:

$$(u, v, w) = \frac{(u', v', w')}{(\Omega L)}, \; (r, z) = \frac{(r', z')}{L}$$

$$t = t'\Omega, \; p = \frac{p'-p_e'}{\rho_0 \Omega^2 L^2}, \; T = \frac{g\alpha\,(T'-T_e')}{\Omega^2 L}$$

$$(h_r, h_\theta, h_z) = \frac{(h_r', h_\theta', h_z')}{H_0}.$$

Where p_e' and T_e' denote the equilibrium values of pressure and temperature respectively.

The boundary conditions are:

$$u=w=T=\frac{dT}{dz}=h_r=h_\theta=h_z=0 \text{ at } z=\pm 1$$

and $v=r$ at $z=1$, $v=-r$ at $z=-1$.

The equations are solved assuming solutions of the Von-Karman type:

$$\psi\,(t, r, z)=r\phi\,(t, z), \; h_r\,(t, r, z)=rg_r\,(z, t)$$

$$v\,(t, r, z)=rV\,(t, z), \; h_0\,(t, r, z)=rg_\theta\,(z, t)$$

$$T=T_0\,(z, t)+r^2T_1, \; h_z\,(t, r, z)=g_z\,(t, z)$$

where ψ denotes the stream function defined by:

$$u=\frac{\partial\psi}{\partial z}, \; w=-\frac{1}{r}\frac{\partial}{\partial r}(r\psi).$$

3. Solution and discussion

Taking Laplace Transform of eqs (1) to (8) and solving them subject to the transformed boundary conditions we get

$$\bar{\phi}=\sum_{j=1}^{4} A_j\,(\cosh m_j z-\cosh m_j) + \frac{2T_1\,Ecd}{\sigma s^2\,(c^2+4d^2)}\left(\frac{\cosh m_5 z}{\cosh m_5}-1\right). \qquad (9)$$

$$\bar{V}=i\sum_{j=1}^{4}(-1)^{\frac{(j-1)(j-2)}{2}}A_j\, m_j \sinh m_j z$$

$$+\frac{4T_1\, Ed^2}{\sigma s^2(c^2+4d^2)}\,\frac{m_5\cosh m_5 z}{\cosh m_5} \tag{10}$$

Where $m^2_{1,2}$ are the roots of

$$(E_m p-s)(Ep-s)-S^2p+2i(E_m p-s)=0 \tag{11}$$

$m^2_{3,4}=m^{*2}_{1,2}$ (* denotes complex conjugate treating s real)

$$A_3=A_1^*,\ A_4=A_2^*,\ m_5=\sqrt{\frac{s\sigma}{E}}$$

A_1, A_2 are known functions of m_1, m_2, s and other parameters.

$c=-(S^2\sigma/E)+(\sigma-1)ds$

$d=(E_m\sigma/E)-1$ and T_1 is determined by the condition

$(d\bar{T}_o/dz)=0$, at the boundaries.

We have not given here the expressions for $\bar{T}_o$, g_r etc, as they are long.

The determination of the functions ϕ, V etc. involves the evaluation of the complex inversion integrals according to Cauchy's Residue Theorem. Therefore, we have to determine the singularities of the integrands. We note that none of the singularities is a branch point as all integrands are even functions of m_1, m_2 etc. The singularity occurring at $s=0$, contributes to the steady state which is discussed in the next section. The other singularities are

$$s=-(n+\tfrac{1}{2})\pi^2E/\sigma,\ n=0,\ \pm1,\ \pm2, \tag{12a}$$

$$s=-n^2\pi^2/E_m \quad n=\pm1,\ \pm2, \tag{12b}$$

$$s=\pm 2i \tag{12c}$$

and at the roots of

$$(E_m m_1^2-s)\, m_1 \sinh m_1 \cosh m_2$$

$$-(E_m m_2^2-s)\, m_2 \sinh m_2 \cosh m_2=0 \tag{13}$$

The singularities in (12a, b) correspond to an infinite number of decaying modes arising due to viscosity and magnetic diffusion, while those in (12c) give rise to oscillatory modes with frequency equal to twice the angular velocity of the frame of reference.

As for the roots of (13) we restrict to the case of weak magnetic field ($S^2 \ll 1$). In this approximation we have

$$s \simeq -\left\{E\tau_{n1} + \tau_{n1}^2 S^2 (E_m - E)/\lambda_1^2\right\} \pm 2i \left\{1 + \tau_n S^2/\lambda_1^2\right\} \tag{14}$$

$$s \cong \left\{-E_m \tau_{n2} + \tau_{n2}^2 S^2 (E - E_m)/\lambda_2^2\right\} \pm 2i \left\{1 + \tau_{n2} S^2/\lambda_2^2\right\}$$

Where $\tau_{n1} = n^2\pi^2$, $\tau_{n2} = (n+\frac{1}{2})^2\pi^2$, $n = 0, \pm 1, \pm 2,$ (15)

$$\lambda_{1,2}^2 = 4 + (E - E_m)^2 \tau_{n1,2}^2.$$

The set of roots in (14) and (15) once again correspond to infinite number of oscillatory modes with decaying amplitudes.

The modes given by (14) are excited by the impulsive angular momentum applied initially and are dampened by viscosity while those given by (15) correspond to the hydromagnetic oscillations induced by rotation, modified by viscosity and dampened by resistivity. From (14) it is clear that a transverse magnetic field always enhances the frequency of these modes while it enhances or diminishes the rates of decay according as $\mu_e \sigma_e \nu \lesseqgtr 1$.

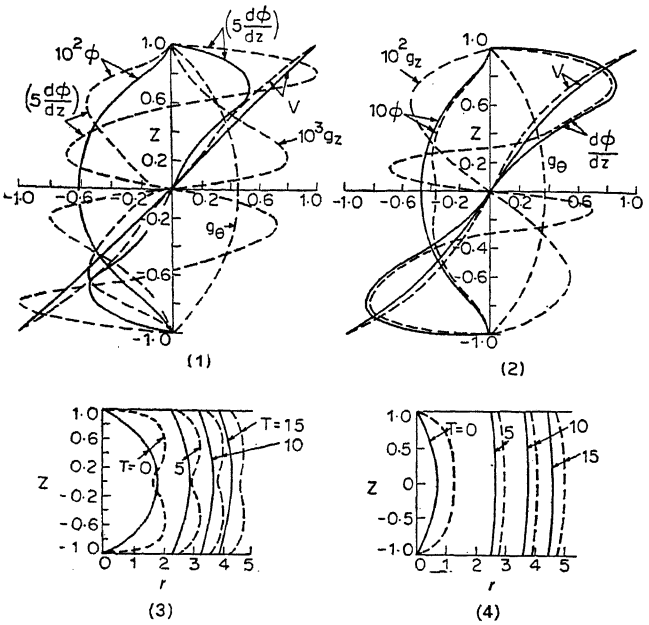
We further note that the equation determining $m_{1,2}^2$ is essentially the same as the one obtained by Loper and Benton (1969) and hence the thickness of the Ekman-Hartmann layer is unaltered by stratification.

4. Steady state solution

The steady state solution can be easily obtained by finding the contribution to the inversion integrals from the singularity at $s=0$. From the steady expressions for velocity and temperature, which are not given here as they are lengthy, it is seen that stratification influences temperature with or without the radial heat flux while it affects velocity only if the radial heat flux does not vanish. Figures 1 to 4 show a comparative picture, between the magnetic and nonmagnetic cases in the velocity and temperature for the two values of $f_R = 0{\cdot}01$ and $0{\cdot}1$. The other parameters are fixed at values:

$$S^2 = 0{\cdot}2,\ E_m = 1,\ \sigma = 0{\cdot}5,\ E = 0{\cdot}1.$$

From figures 1 and 2 we conclude that, in the presence of stratification, a transverse magnetic field has always a deceleration effect on the fluid motion. While the decelaration is not large in the azimuthal direction it is of such an appreciable magnitude in the radial and axial directions that a reversal of radial velocity takes



Figures 1 and 2. Velocity and magnetic field distributions. —Non magnetic case ----Magnetic case

1. $f_R / \sigma = 0{\cdot}02$ **2.** $f_R / \sigma = 0{\cdot}2$.

Figures 3 and 4. Isotherms. —Non magnetic case. - - - Magnetic case.

3. $f_R / \sigma = 0{\cdot}02$. **4.** $f_R / \sigma = 0{\cdot}2$.

place across two planes symmetrically situated with respect to the midplane. The figures also show that the influence of magnetic field increases with stratification.

The curves in figures 3 and 4 which represent the isotherms show that the effect of magnetic field is to shift the isotherms away from the axis in the radial direction. From figure 3 we observe that the inhibiting influence of the magnetic field becomes very appreciable that the curvature of the isotherm changes near the core thereby showing that the flow of heat away from that region, while in the non-magnetic case the direction of heat flow is always towards the core region. Comparison of figures 3 and 4 show that the influence of magnetic field on the temperature also increases with the stratification. Finally, we note that the steady state moment coefficient

$$c_M = E\pi \left(\frac{dv}{dz}\right)_{z=\pm 1}$$

is also diminished in magnitude by the effect of magnetic field as seen by comparing the shapes of the azimuthal velocity curves near the boundary.

5. Conclusion

From the above discussion it is concluded that neither stratification nor temperature gradient along the disks affects the rate of growth of the Ekman-Hartman

boundary layer thickness and its steady state value is as obtained by Loper and Benton (1969). The inhibiting influence of magnetic field on the spin of the fluid is found to be more pronounced in the presence of stratification as compared with the homogeneous fluid. The transient solution exhibits two infinite sets of decaying modes arising from viscosity and resistivity and another set of oscillatory modes with decaying amplitudes excited by the initial angular momentum whose frequencies are affected by magnetic field but not by stratification.

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Steady flow of a viscous liquid in a porous elliptic tube

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Abstract. Flow of a viscous-liquid in a porous tube of elliptic cross-section is studied using the generalized momentum equation. As a particular case, flow of the liquid in a tube of a circular cross-section is obtained. It is observed that the classical Darcian effect is realized only in a core very near to the axis of the tube while the non-Darcian phenomenon is felt predominantly near the boundary of the tube.

Keywords. Porous; highly porous; permeability; Mathieu function.

1. Introduction

Flow of a viscous liquid in a porous medium is of importance in the study of percolation through soils in hydrology, petroleum industry and in agricultural engineering. Henry Darcy had observed while studying flow of water through sand filters that the flow rate of water is proportional to the difference in head of water across the filter. Subsequently, many experiments were conducted to study the flow of various fluids through different types of porous solids.

Brinkman (1947), proposed a generalized Darcy's law to study flow through highly porous media.

$$0 = -\nabla \mathbf{p} - (\mu/k)\mathbf{V} + \mu \nabla^2 \mathbf{V}$$

where $\mathbf{V}$, $\mathbf{p}$ represent velocity and pressure fields, μ is viscosity coefficient of fluid and k is the permeability of the medium. Later Tam (1969) derived analytically the same equation to study flow past spherical particles at low Reynold's number.

The generalized law is found useful in the study of flow in highly porous media such as pappus of dendelion and fibres. Yamamoto (1971, 1973) examined flow past porous bodies applying the generalized law.

We observe that the classical Darcian effect is seen only in a core very near to the axis of the tube and at the boundary the non-Darcian effect is felt predominantly.

2. Basic equations and solution

Brinkman (1947) proposed the basic equation for the slow steady flow of a viscous liquid in a porous medium as a generalization to the classical Darcy's law.

$$0 = -\nabla p + \mu \nabla^2 \mathbf{v} - (\mu/k)\mathbf{v} \qquad (1)$$

together with the equation of continuity

$$\text{div } \mathbf{v}=0 \tag{2}$$

Let (x, y, z) be the rectangular co-ordinate system such that Z-axis lying along the length of the tube with impermeable boundary T. The velocity components in x, y, z directions are taken to be 0, 0, w (x, y) respectively. The equation of continuity is satisfied with the choice of the velocity and the equation of motion becomes

$$0=-\frac{\partial p}{\partial z}+\mu\nabla^2 w-\frac{\mu}{k}w \tag{3}$$

with $w=0$ on T, the boundary of the tube. (4)

If the pressure gradient $-\partial p/\partial z=\rho\alpha$, is a constant, we re-write eq. (3) as

$$\nabla^2\phi-\frac{\phi}{k}=0 \tag{5}$$

where

$$\phi=w-\frac{\alpha k}{\nu},\ \nu=\mu/\rho \tag{6}$$

and $$\phi=-\frac{\alpha k}{\nu} \text{ on } T. \tag{7}$$

The transformation of eq. (5) into elliptic co-ordinates (ξ, η) can be done by taking $x+iy=C\cosh(\xi+i\eta)$ as in Mc Lachlan (1964, pp. 170–173), which gives

$$\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}-2q(\cosh 2\xi-\cos 2\eta)\phi=0 \tag{8}$$

where $q=(1/k)C^2/4$ and $C^2=a^2-b^2$ with $a=C\cosh\xi$ and $b=C\sinh\xi$ as the half of the major and minor axes of the elliptic cross-section T.

Let $\phi(\xi, \eta)=f(\xi)\,g(\eta)$ in the above equation then we get,

$$\frac{d^2f}{d\xi^2}-[\lambda^{(2n)}+2q\cosh 2\xi]f=0 \tag{9}$$

$$\frac{d^2g}{d\eta^2}+[\lambda^{(2n)}+2q\cos 2\eta]\,g=0. \tag{10}$$

Here $\lambda^{(2n)}$ is a separation constant. The flow is symmetric about the axes of the ellipse and is periodic with period π in η. The solutions of eqs (9) and (10) are given by Mc Lachlan (1964, pp. 21, 27)

as $g=Ce_{2n}(\eta, -q)$

and $f=Ce_{2n}(\xi, -q)$.

Hence ϕ is given by

$$\phi = \sum_{n=0}^{\infty} P_{2n} Ce_{2n}(\xi, -q)\, Ce_{2n}(\eta, -q) \tag{11}$$

where $Ce_{2n}(\eta, -q) = (-1)^n \sum_{n=0} (-1)^r A_{2r}^{(2n)} \cos 2r\eta.$ (12)

$$Ce_{2n}(\xi, -q) = (-1)^n \sum_{n=0}^{\infty} (-1)^r A_{2r}^{(2n)} \cosh 2\xi r \tag{13}$$

and the coefficients $A_{2r}^{(2n)}$ are functions of q. The constants P_{2n} are determined using the boundary condition on $\xi = \xi_0$.

$$-a\frac{k}{\nu} = \Sigma P_{2n}\, Ce_{2n}(\xi_0, -q)\, Ce_{2n}(\eta, -q). \tag{14}$$

Multiplying both sides of eq. (14) by $Ce_{2n}(\eta, -q)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation of Mc Lachlan (1964) we get

$$P_{2n} = \frac{ak}{\nu} \frac{(-1)^{n+1}\, 2\pi\, A_0^{2n}}{Ce_{2n}^2(\xi_0, -q)\, I_{2n}} \tag{15}$$

where

$$I_{2n} = \int_0^{2\pi} Ce_{2n}^2(\eta, -q)\, d\eta \tag{16}$$

Hence $w = \phi + \frac{ak}{\nu}$

$$= \frac{ak}{\nu}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, 2\pi\, A_0^{(2n)}}{Ce_{2n}(\xi, -q) I_{2n}}\, Ce_{2n}(\xi, -q)\, Ce_{2n}(\eta, q)\right] \tag{17}$$

Case I

when $k \to \infty$, $q \to 0$ (Mc Lachlan 1964 p. 15)

$$Ce_0(\eta, -q) 1 + (\tfrac{1}{2}) q \cos 2\eta \tag{18}$$

$$Ce_{2n}(\eta, -q) \cos 2\eta + \{(-\tfrac{1}{4})(1/12) \cos 4\eta)\}\, q \tag{19}$$

$$Ce_{2n}(\xi, -q) = Ce_{2n}(i\xi, -q) \tag{20}$$

$$A_0^{(0)} = 1,\ A_0^{(2)} = 0\ A_0^{(2n)} = 0 \text{ for all } n > 2 \tag{21}$$

$$P_0 = \frac{aK}{\nu}(1-(\tfrac{1}{2})q\cosh 2\xi_0) \tag{22}$$

$$P_2 = \frac{aK}{\nu}\left(\frac{q}{2\cosh 2\xi_0}\right), p_{2n} = 0: \quad n > 2 \tag{23}$$

and velocity w becomes

$$w = \frac{a}{2\nu}\frac{a^2b^2}{(a^2+b^2)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{24}$$

which is the same as the velocity of a viscous liquid in an elliptic tube when no resistance is offered by the medium.

Case II

when k is very small q is very large

$$A_0^{(0)} = 1, \; A_0^{(2n)} \simeq 0 \text{ for } n > 1 \tag{25}$$

and from Mc Lachlan (1964 p. 230)

$$Ce_{2n}(\xi, -q) \simeq \frac{(-1)^n C_{2n}}{2^{2n-\frac{1}{2}}(i\sinh)^{\frac{1}{2}}}\cosh\left[2q^{\frac{1}{2}}\cosh\xi\right.$$

$$\left.-(4n+1)\tanh^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{i\xi}{2}\right)\right\}\right] \tag{26}$$

$$C_{2n} = \frac{(-1)^n 2^{2n-\frac{1}{2}}}{A_0^{(2n)}(\pi/\sqrt{q})^{\frac{1}{2}}} Ce_{2n}(0)\, Ce_{2n}(\pi/2);$$

$$C_{2n} \simeq 0 \text{ for } n > 1 \text{ for large } |q| \tag{27}$$

$$\phi \simeq P_0\, Ce_0(\xi, -q)\, Ce_0(\eta, -q)$$

$$\simeq -\frac{ak}{\nu}\frac{\cosh(\frac{1}{2}\xi_0)}{\cosh(\frac{1}{2}\xi)}\exp(-2/q^{\frac{1}{2}})(\cosh\xi_0 - \cosh\xi).\quad Ce_0(\eta, -q). \tag{28}$$

Hence w is given by (Mc Lachlan 1964, p. 388)

$$w = \frac{ak}{\nu}\left[1 - \frac{\cosh\frac{1}{2}\xi_0}{\cosh\frac{1}{2}\xi_0} Ce_0(\eta, -q)\exp(-d/k^{1/2})\right] \tag{29}$$

where $d = c(\cosh\xi_0 - \cosh\xi)$

he tube is of circular cross-section

. circle $c \to 0$, $\xi_0 \to \infty$ such that $a = \cosh \xi_0$.
ıs to a circle of radius a and also (Mc Lachlan 1964 p. 367).

$$, -q) \to 2^{-1/2} \tag{30}$$

$$\eta - q) \to \cos 2n\eta,\ n > 1 \tag{31}$$

$$\xi, -q) \to P'_{2n}.I_{2n}(K_1 \eta):\ n > 0 \tag{32}$$

$$\xi_0 - q) \to P'_{2n} I_{2n}(K_1 a):\ n > 0 \tag{33}$$

$$\to 2^{-1/2} \tag{34}$$

$$\to 0 : n > 1 \tag{35}$$

$$0 \tag{36}$$

$$, r = \cosh \xi = (x^2 + y^2)^{1/2} \tag{37}$$

hese limiting values we get

$$\frac{k}{\nu}\left[1 - \frac{I_0(r/k^{1/2})}{I_0(a/k^{1/2})}\right] \tag{38}$$

: as that obtained by Pattabhiramacharyulu (1976).

he second term in eq. (36) damps out as d, the distance from the increases while the first term remains almost constant ak/ν, which at obtained under the classical Darcy's law (Mushakat 1937).

$$-\nabla \mathbf{p} - \frac{\mu}{k}\mathbf{V}.$$

arcian effect is seen to exist predominantly near the boundary of the sical Darcian effect is realized only in a core near to the axis of the

omenon is observed by Sexl (1930) in the study of pulsating flow cous fluid in a circular tube and by Pattabhiramacharyulu (1976) orous circular tube.

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Phase space methods and the Hamilton-Jacobi form of dynamics

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Abstract. A general analysis of the Hamilton-Jacobi form of dynamics motivated by phase space methods and classical transformation theory is presented. The connection between constants of motion, symmetries, and the Hamilton-Jacobi equation is described.

Keywords. Classical dynamics; phase-space method; Hamilton-Jacobi theory.

1. Introduction

The formulation of classical dynamics based on the Hamilton-Jacobi (H-J) equation has played an important role in the development of quantum mechanics. In the days of the old quantum theory it was used most effectively by Sommerfeld, Wilson, Epstein and Einstein among others to extend the Bohr quantization condition, originally formulated for the hydrogen atom, to more complex systems (Sommerfeld [14]; Wilson [19]; Epstein [6]; Einstein [5]; see also Born [1], Sommerfeld [15]. Later on Schrödinger [13] used the H-J equation as the starting point in his construction of wave mechanics based on de Broglie's theory of phase waves. Conversely, once the Schrödinger wave equation had been established, the H-J equation was recovered as the first step in solving the wave equation in the quasi-classical WKB approximation (Wentzel [17]; Kramers [7]; Brillouin [2]).

After the advent of quantum mechanics the study of invariance and symmetry properties of dynamical systems took on great importance. The linear structure of quantum mechanics expressed by the superposition principle gave to this study great elegance and simplicity, and enabled one to draw extensively upon the mathematical theory of group representations. At least partly due to the fact that for some time now quantum mechanics has been much more thoroughly studied and taught than classical mechanics, there is the feeling that the mathematical description of symmetry is easier to grasp in the quantum context than in the classical one. However it must be admitted that the mathematical description of symmetry is an important part of the formalism of classical mechanics. Classical descriptions of symmetry and invariance of course do exist, both in the Lagrangian form of mechanics and in the Hamiltonian phase space form. The intimate connection between symmetries and conservation laws expressed by the Noether theorems is an important aspect of the Lagrangian formulation (Noether [9]). And the realisation of symmetry operations as canonical transformations on phase space is an important aspect of the Hamiltonian formulation.

Given all this it would seem somewhat surprising that there appears to be no easily accessible discussion of the H-J form of dynamics which analyses the action of symmetry operations directly on the H-J equation and its solutions. (Several classic treatments of the H-J form of dynamics are, of course, available; see, for instance, Nordheim and Fues [10] Whittaker [18]; Lanczos [8]). The corresponding quantum discussion in which symmetry operations are implemented by the action of unitary transformations on solutions to the Schrödinger equation is of course well known. But a discussion of how concepts and methods natural to the phase space description of dynamics can be adapted to the H-J description seems unavailable. The concept of the Poisson Bracket (PB) among phase space functions, and the generation of canonical transformations by given phase space functions which may in particular be constants of motion, are among the important elements of the phase space description of dynamics; and it would be instructive to see how these are related to the H-J equation and its solutions. There is also another point of view from which it is illuminating to study the H-J theory, which is the following. It is well known that an individual solution of the H-J equation describes a family of solutions of the Hamiltonian equations of motion, constructed in a special way. It is these classical families that stand in correspondence with individual solutions of the Schrödinger equation, which of course describe single states of motion in quantum mechanics. At the kinematic level a single function on configuration space (which in a well known way generates one of the above mentioned classical families of trajectories) is analogous to one Schrödinger wavefunction or more generally one state in quantum mechanics. For state vectors many characteristically quantum mechanical properties obtain. For instance, a state vector can be a simultaneous eigenvector of two distinct operators only if it is also an eigenvector of their commutator corresponding to the eigenvalue zero. Again a given state vector can always be expanded as a linear combination of, say, energy eigenstates, and there is a straightforward way to isolate individual terms in this expansion. All these features of quantum mechanics turn out to possess analogues in the H-J form of classical mechanics, some being better known than others. The purpose of this paper is to provide a study of the H-J theory from these points of view.

The H-J equation forms an important component of a highly developed part of classical mathematics having deep connections with the calculus of variations and with systems of ordinary differential equations (Caratheodory [3]; Rund [11]). However a discussion of it along the lines mentioned above seems unavailable.

The material of this paper is arranged as follows. In section 1, some useful geometrical ideas in phase space are introduced. The essential point is to associate a surface in phase space with every configuration space function in a special way and with certain intrinsic canonical invariant features. The action of canonical transformations on these surfaces and evaluation of PB's on them are analysed, permitting a simple discussion of simultaneous H-J equations. Section 2 gives a detailed geometrical interpretation of solutions of the time-dependent H-J equation, and the action of symmetry transformations on the manifold of solutions. The contrast between values of a dynamical variable and the transformations it generates is clearly exhibited. In section 3 the relations between solutions of the time dependent and time-independent H-J equations is analysed motivated by similar situations in quantum mechanics. One is thereby led to a better understanding of a well known method of passing from one equation to the other. Section 4 introduces very

briefly the notion of H-J equations on a Lie group. Section 5 contains concluding remarks. The appendix summarises some useful properties of canonical transformations and Lagrange brackets in phase space, and the general mathematical techniques for building up solutions of the H-J equations.

2. Geometry of phase space surfaces

The usual geometrical pictures that are constructed to aid in the understanding of the H-J equation are confined to configuration space. They are based on Huyghen's principle in optics and the analogy between optics and mechanics discovered by Hamilton (see, for instance, Lanczos [8]; Rund [11]). We shall find it more useful and suggestive to work in phase space since that makes the action of canonical transformations easier to visualise.

We consider a classical dynamical system with n degrees of freedom and a $2n$ dimensional phase space with canonical coordinates q_r, p_r, $r=1, 2, \ldots, n$. The same system of coordinates will be used throughout the discussion, so that canonical transformations will always be interpreted in the active sense as arising from invertible mappings of the phase space onto itself. The condition for such a mapping to be canonical is given in the appendix (eq. (A.1)).

Let M be an m-dimensional hypersurface embedded in the $2n$ dimensional phase space, and let the real independent variables u_α, $\alpha = 1, 2, \ldots, m$, give a parametrisation of M. The points of M then appear as $q_r(u)$, $p_r(u)$. The Lagrange brackets (LB) of the u_α with one another are defined as

$$[u_\alpha, u_\beta] = \frac{\partial q_r(u)}{\partial u_\alpha}\frac{\partial p_r(u)}{\partial u_\beta} - \frac{\partial q_r(u)}{\partial u_\beta}\frac{\partial p_r(u)}{\partial u_\alpha} \tag{1}$$

[Summation over repeated indices is understood]. With these LB's as elements we can set up an $m \times m$ real antisymmetric matrix. We assume for simplicity that this matrix has constant rank over M, and call it the symplectic rank of M. It is easily verified that the symplectic rank is independent of the choice of parameters u_α. Let a canonical transformation now be given. Applied to the points of M it leads to another m-dimensional hypersurface, M' say, for which the u_α again serve as parameters. The point $q_r(u)$, $p_r(u)$ on M is carried by the canonical transformation to $Q_r(u)$, $P_r(u)$ on M'. It now follows from the invariance of LB's under canonical transformations (proved in the appendix) that M' has the same symplectic rank as M. Thus the symplectic rank of a hypersurface in phase space is an intrinsic canonical invariant property of the hypersurface.

Let us now consider the case that M has zero symplectic rank. The appendix shows that the functions $q_r(u)$, $p_r(u)$ must then be such that we have

$$p_r(u)\frac{\partial q_r(u)}{\partial u_\alpha} = \frac{\partial s(u)}{\partial u_\alpha} \tag{2}$$

for some function $s(u)$. If $m < n$, knowledge of $q_r(u)$ and $s(u)$ does not suffice to determine $p_r(u)$. Let us assume next that $m=n$ and that the u_α can be taken to be

the position variables q_r. In that case we can set $s(q)$ in place of $s(u)$ in eq. (2) and see that on M the momenta p_r are functions of the q_r:

$$p_r = \frac{\partial s(q)}{\partial q_r}. \tag{3}$$

Thus an n-dimensional hypersurface of vanishing symplectic rank over which the q_r are independent variables is uniquely determined by any given function $s(q)$ on configuration space; and conversely such a hypersurface determines an $s(q)$ up to an additive constant. We shall use the notation $\Gamma[s(q)]$ for such hypersurfaces in phase space, additional arguments of s being inserted when necessary. Occasionally the arguments of s will be omitted if no confusion is likely to arise.

Let $f(q, p)$ be a given phase space function. On restricting the arguments of f to a $\Gamma[s]$ we produce a function on configuration space; this is achieved by substituting for p_r in f from eq. (3). We shall describe this process by which one passes from a phase space function to a configuration space function, given an $s(q)$, with this notation:

$$f\left(q, \frac{\partial s(q)}{\partial q}\right) \equiv (f(q, p))_s. \tag{4}$$

It is useful to know the relationship between the PB operation and this restriction operation. If $g(q, p)$ is another phase space function, the PB of f with g is a third phase space function given by

$$\{f, g\} = \frac{\partial f}{\partial q_r}\frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial g}{\partial q_r}. \tag{5}$$

Naturally the restriction of $\{f, g\}$ to $\Gamma[s]$ cannot be expressed in terms of the restrictions of f and g alone; however with the help of eq. (3) one easily finds the very important result

$$(\{f, g\})_s = \frac{\partial (f)_s}{\partial q_r}\left(\frac{\partial g}{\partial p_r}\right)_s - \left(\frac{\partial f}{\partial p_r}\right)_s \frac{\partial (g)_s}{\partial q_r}. \tag{6}$$

We will use this formula repeatedly in what follows.

With the concepts introduced so far, let us examine the action of canonical transformations on hypersurfaces of the type $\Gamma[s]$. We shall only be interested in infinitesimal transformations generated by given functions $A(q, p)$ on phase space. The explicit formula for such canonical transformations is given in eq. (A.2) of the appendix. A concise notation for the infinite power series occurring in the transformation is achieved if we introduce a partial differential operator A^* associated with the function A (the asterisk is always used in this sense throughout the paper, and never to denote complex conjugation):

$$A^* = \frac{\partial A}{\partial q_r}\frac{\partial}{\partial p_r} - \frac{\partial A}{\partial p_r}\frac{\partial}{\partial q_r} \cdots \tag{7}$$

The action of A^* on a function $B(q, p)$ then produces the PB $\{A, B\}$. Then, the canonical transformation with parameter ϵ and infinitesimal generator $A(q, p)$ takes the form

$$Q_r(q, p, \epsilon) = \exp(\epsilon A^*)\, q_r,\ P_r(q, p, \epsilon) = \exp(\epsilon A^*)\, p_r \tag{8}$$

The transformation itself may be denoted by the symbol $\exp(\epsilon A^*)$. Let us now apply this canonical transformation to the points comprising a given hypersurface $\Gamma[s]$. Since we shall be interested only in terms up to at most second order in the (small) parameter ϵ, we may assume that the image of $\Gamma[s]$ also permits the use of the q_r as independent parameters over it; thus this image must be $\Gamma[s']$ for some $s'(q)$. We can in principle get s' using eq. (A.3) of the appendix: if Q_r, P_r is the image in $\Gamma[s']$ of q_r, p_r in $\Gamma[s]$, then

$$P_r dQ_r = d\left[s(q) + \epsilon(A - p_r\{q_r, A\})_s - \frac{\epsilon^2}{2}(\{A, p_r\{q_r, A\}\})_s \ldots\right]. \tag{9}$$

By expressing the function within square brackets as a function of Q rather than q, we discover the functional form of s'. After some algebra we get the result

$$s'(q) = s(q) + \epsilon(A)_s + \frac{\epsilon^2}{2}\frac{\partial(A)_s}{\partial q_r}\left(\frac{\partial A}{\partial p_r}\right)_s \ldots \tag{10}$$

(One recognises that this is just the solution of the H-J equation corresponding to A and with ϵ as parameter). So we can also write

$$\exp(-\epsilon A^*)\,\Gamma[s] = \Gamma\left[s + \epsilon(A)_s + \frac{\epsilon^2}{2}\frac{\partial(A)_s}{\partial q_r}\left(\frac{\partial A}{\partial p_r}\right)_s \ldots\right]. \tag{11}$$

The negative sign in the exponent is introduced on purpose with a view to securing the Baker-Campbell-Hausdorff formula in its usual form later on. We must note two important points in this connection. The first is that whereas in eq. (8) A^* is to be interpreted as a partial differential operator that acts on any function of q, p that may stand to its right, in eq. (11) it is not to be interpreted literally in this sense but only symbolically. In fact the action of the symbol $\exp(-\epsilon A^*)$ on a hypersurface Γ is completely defined by eq. (11). The second is that eq. (11) does not tell us how each point on $\Gamma[s]$ is affected by the canonical transformation but only how the function s is affected; thus it only specifies the image of $\Gamma[s]$ as a whole.

In the remainder of this section we use the definitions and results obtained thus far to discuss some of the kinematic features associated with the H-J form of dynamics. It will be seen that these are very similar to well known features of quantum mechanics. The H-J equation *per se* is taken up in the succeeding sections. Let us ask first under what conditions a hypersurface $\Gamma[s]$ is invariant under a canonical transformation $\exp(-\epsilon A^*)$. Working only to the order ϵ, we see from eq. (11), that this happens if and only if

$$(A)_s \equiv A\left(q, \frac{\partial s}{\partial q}\right) = a, \tag{12}$$

where a is some constant. This equation can be called the H-J equation (time-independent!) corresponding to the function $A(q, p)$. Thus $\Gamma[s]$ is invariant under exp $(-\epsilon A^*)$ if and only if s obeys the H-J equation corresponding to A for some value of the constant a. This can be taken to be the analogue of the following statement in quantum mechanics: A (pure) physical state is unaffected by the unitary transformation generated by a given hermitian operator if and only if the corresponding state vector is an eigenvector of the operator in question. The fact that adding a constant to s does not change $\Gamma[s]$ corresponds exactly to the fact that a physical state is unaltered when the state vector is multiplied by a phase factor. [Incidentally when eq. (12) is obeyed the ϵ^2 term in eq. (11) is identically zero; this will persist in higher orders]. When eq. (12) holds, the structure of $\Gamma[s]$ can be described in this way: Clearly the orbits of points in $\Gamma[s]$ under the one-parameter family (group) of canonical transformations exp (ϵA^*) lie entirely in $\Gamma[s]$. Thus $\Gamma[s]$ can be reconstructed if one is given a suitable $(n-1)$-dimensional subsurface of it, a cross-section of the orbits, to all points of which one applies the canonical transformation exp (ϵA^*) for all real ϵ. This is just the way in which solutions of the time-independent H-J equation can be generated, as described in the appendix.

Suppose next that $\Gamma[s]$ is invariant under the canonical transformations generated by $A(q, p)$ as well as under those generated by some $B(q, p)$. Thus s must obey two H-J equations simultaneously:

$$(A)_s = a, \quad (B)_s = b. \tag{13}$$

Using this in the PB formula eq. (6) we see immediately that

$$(\{A, B\})_s = 0. \tag{14}$$

Thus s obeys the H-J equation corresponding to the phase space function $\{A, B\}$ with the constant on the right being zero. (This is a known result; see for instance Caratheodory [3] § 58). This is the analogue to the statement in quantum mechanics that if a state vector is simultaneously an eigenvector of two distinct operators, then it is also an eigenvector of their commutator with eigenvalue zero. Conversely, if the PB $\{A,B\}$ is non zero throughout phase space, which happens if A and B form a canonically conjugate pair of functions, no function $s(q)$ can obey simultaneously the H-J equations corresponding to A and to B.

2. Symmetry transformations and the H-J equation

We analyse in this section the action of symmetries of a dynamical system on the solutions of the H-J equation. For this purpose we first build up a geometrical picture for the solutions themselves. Let the possibly explicitly time-dependent Hamiltonian be $H(q, p, t)$ and let $S(q; t)$ be a solution of the H-J equation

$$(H(q, p, t))_S + \frac{\partial S(q; t)}{\partial t} = 0. \tag{15}$$

For each time t we construct the corresponding hypersurface $\Gamma\,[S(q;t)]$ in phase space. This hypersurface then moves with time, always preserving its character of being n-dimensional and of having vanishing symplectic rank. That S obeys eq. (15) can now be interpreted in this way: the hypersurface at time $t+\delta t$ coincides with the result of applying the infinitesimal canonical transformation with generator $-H(q,p,t)\,\delta t$ to the hypersurface at time t. This is entirely natural, but it is an interesting application of eq. (11). Retaining only terms up to first order in δt, we have:

$$\begin{aligned}
&\exp\,(\delta t\, H(q,p,t)^*)\;\Gamma\,[S(q;t)\,] \\
&\quad = \Gamma\,[S(q;t) - \delta t\,(H(q,p,t))_S] \\
&\quad = \Gamma\left[S(q;t) + \delta t \frac{\partial S(q;t)}{\partial t}\right] \\
&\quad = \Gamma\,[S(q;t+\delta t)\,]. \qquad (16)
\end{aligned}$$

Use of eq. (11) was followed by use of eq. (15). It is true that motion according to Hamilton's equations from time t to $t+\delta t$ is an infinitesimal canonical mapping of phase space onto itself but that is not the content of eq. (16). Rather we must view eq. (16) as expressing the following property of S: If S is any solution whatsoever of the H-J eq. (15), the evolution in time of $\Gamma\,[S(q;t)\,]$ as determined by the explicit time dependence of S coincides with the canonical evolution as determined by eq. (11) with $H(q,p,t)$ as generator. This interpretation is well known. Equation (16) gives a geometrical picture of it.

Next suppose a phase space function $G(q,p,t)$ is given, which happens to be a constant of motion:

$$\{G,H\} + \frac{\partial G}{\partial t} = 0. \qquad (17)$$

In the transformation theory of dynamics, this is usually interpreted thus: under the (time-dependent) canonical transformation generated by G the Hamiltonian preserves its functional form. We wish to connect this with the properties of solutions of the H-J equation. Let us recall first the analogous situation in quantum mechanics. Let a quantum system have a Hamiltonian operator $\hat{H}(t)$ and let $\hat{G}(t)$ be a hermitian constant of motion, either or both possessing explicit time dependence (we work in the Schrödinger picture of quantum mechanics):

$$-i[\hat{G}(t),\ \hat{H}(t)\,] + \frac{\partial\,\hat{G}(t)}{\partial t} = 0. \qquad (18)$$

If $|\,\psi(t)\rangle$ is any solution of the Schrödinger equation

$$i\frac{d}{dt}\,|\,\psi(t)\rangle = \hat{H}(t)\,|\,\psi(t)\rangle, \qquad (19)$$

and one defines a unitary operator $U(\epsilon; t)$ by

$$U(\epsilon; t) = \exp\left(i \epsilon \hat{G}(t)\right),$$

then one finds that

$$|\psi'(t)\rangle = U(\epsilon; t)\,|\psi(t)\rangle$$

is also a solution of the Schrödinger equation. It is very easy to show this for infinitesimal ϵ, and with a little operator algebra for any ϵ. Thus the (possibly time dependent) unitary transformation generated by a (possibly explicitly time dependent) constant of motion $\hat{G}$ maps any possible state of motion onto another such. We seek the corresponding result within H-J theory. Given $G\ (q, p, t)$ obeying eq. (17) and any solution $S(q; t)$ of eq. (15), let us set up the time dependent hypersurface $\Gamma[S(q; t)]$ and apply to it an infinitesimal canonical transformation with parameter ϵ and generator G taken at time t; from eq. (11) we get

$$\exp\left(-\epsilon\, G\,(q,p,t)^*\right) \Gamma\,[S\,(q;t)] = \Gamma\,[S'\,(q;t)],$$

$$S'\,(q;t) = S\,(q;t) + \epsilon\left(G\,(q,p,t)\right)_S. \tag{20}$$

We first confirm by direct calculation that S' too obeys the H-J equation, and then show it geometrically. The time arguments being t throughout, we have:

$$(H)_{S'} - (H)_S = \epsilon \left(\frac{\partial H}{\partial p_r}\right)_S \frac{\partial (G)_S}{\partial q_r}; \tag{21a}$$

$$\frac{\partial S'}{\partial t} - \frac{\partial S}{\partial t} = \epsilon \frac{\partial (G)_S}{\partial t}$$

$$= \epsilon \left(\frac{\partial G}{\partial t}\right)_S + \epsilon \left(\frac{\partial G}{\partial p_r}\right)_S \frac{\partial^2 S}{\partial q_r\, \partial t}$$

$$= \epsilon \left(\frac{\partial G}{\partial t}\right)_S - \epsilon \left(\frac{\partial G}{\partial p_r}\right)_S \frac{\partial (H)_S}{\partial q_r}. \tag{21b}$$

By combining these with the PB restriction formula (6) and with the restriction of eq. (17) to $\Gamma[S(q; t)]$ we see immediately that $S'(q; t)$ is also a solution of the H-J equation. Thus we have the classical result that any constant of motion G brings with it a one-to-one mapping of the manifold of solutions of the H-J equation onto itself. Recalling that, as shown in the appendix, each solution $S(q; t)$ of the H-J equation is determined in a one-one manner by a function $s(q)$ on configuration space, we can also say: the phase space canonical transformation generated by G gives rise to a mapping of configuration space functions onto themselves that preserves the H-J equation.

It is illuminating to give another, more geometrical, derivation of this result. This is where we need to use the second order terms in ϵ appearing on the right hand

side of eq. (11). Let ϵ, ϵ' be two independent small parameters, $A(q, p)$ and $B(q, p)$ two phase space functions, and $\Gamma[s]$ a given hypersurface. We wish to calculate the result of applying two infinitesimal canonical transformations in succession to $\Gamma[s]$. Let us write

$$\exp(-\epsilon' B^*)\exp(-\epsilon A^*)\,\Gamma[s] = \exp(-\epsilon' B^*)\,\Gamma[s_1] = \Gamma[s_2]. \tag{22}$$

Our objective is to relate s_2 to s in the manner of eq. (11) via a single canonical transformation, retaining at all stages terms that are at most of second order in ϵ and ϵ' jointly. We relate s_1 to s, and then s_2 to s_1, and retain the terms of interest to get:

$$s_1 = s + \epsilon (A)_s + \frac{\epsilon^2}{2}\frac{\partial (A)_s}{\partial q_r}\left(\frac{\partial A}{\partial p_r}\right)_s; \tag{23a}$$

$$\begin{aligned} s_2 &= s_1 + \epsilon' (B)_{s_1} + \frac{\epsilon'^2}{2}\frac{\partial (B)_{s_1}}{\partial q_r}\left(\frac{\partial B}{\partial p_r}\right)_{s_1} \\ &\simeq s + \epsilon (A)_s + \frac{\epsilon^2}{2}\frac{\partial (A)_s}{\partial q_r}\left(\frac{\partial A}{\partial p_r}\right)_s + \epsilon'\left[(B)_s + \epsilon\left(\frac{\partial B}{\partial p_r}\right)_s \frac{\partial (A)_s}{\partial q_r}\right] \\ &\quad + \frac{\epsilon'^2}{2}\frac{\partial (B)_s}{\partial q_r}\left(\frac{\partial B}{\partial p_r}\right)_s \\ &= s + (\epsilon A + \epsilon' B)_s + \frac{1}{2}\frac{\partial (\epsilon A + \epsilon' B)_s}{\partial q_r}\left(\frac{\partial (\epsilon A + \epsilon' B)}{\partial p_r}\right)_s \\ &\quad + \frac{1}{2}\epsilon\epsilon'\left[\frac{\partial (A)_s}{\partial q_r}\left(\frac{\partial B}{\partial p_r}\right)_s - \left(\frac{\partial A}{\partial p_r}\right)_s \frac{\partial (B)_s}{\partial q_r}\right] \\ &\approx s + (C)_s + \frac{1}{2}\frac{\partial (C)_s}{\partial q_r}\left(\frac{\partial C}{\partial p_r}\right)_s, \end{aligned}$$

$$C = \epsilon A + \epsilon' B + \tfrac{1}{2}\epsilon\epsilon'\{A, B\}. \tag{23b}$$

Note the use of eq. (6) at the last step but one in the above. We thus have the formula, valid to quadratic terms,

$$\begin{aligned} &\exp(-\epsilon' B^*)\exp(-\epsilon A^*)\,\Gamma[s] \\ &\quad = \exp(-\epsilon A^* - \epsilon' B^* - \tfrac{1}{2}\epsilon\epsilon'\{A, B\}^*)\,\Gamma[s]. \end{aligned} \tag{24}$$

This is the Baker-Campbell-Hausdorff formula for composition of canonical transformations, to lowest nontrivial order. (For more details see, for instance, Sudarshan and Mukunda [16]). On interchanging the roles of A and B, we get a similar formula; and comparing the two we get, again to the second order, the result

$$\begin{aligned} \exp(-\epsilon' B^*)\exp(-\epsilon A^*) &= \exp(-\epsilon\epsilon'\{A, B\}^*)\exp(-\epsilon A^*)\exp(-\epsilon' B^*) \\ &\simeq \exp\left(-\epsilon (A + \epsilon'\{A, B\})^*\right)\exp(-\epsilon' B^*) \end{aligned} \tag{25}$$

valid in the sense of eq. (11) *when both sides are applied to any* $\Gamma[s]$. In this general result, let us take A to be the constant of motion $G(q,p,t)$, B the Hamiltonian $H(q,p,t)$ and $-\epsilon'$ an infinitesimal time interval δt. Because of eq. (17) the combination of A and B on the right hand side of eq. (25) becomes

$$A + \epsilon' \{A, B\} = G(q, p, t + \delta t). \tag{26}$$

If we now apply both sides of eq. (25) to a hypersurface $\Gamma[S(q,t)]$ where S is any solution of the H-J eq. (15) and use eq. (16) as well as the definition of S' given in eq. (20), we get:

$$\begin{aligned} &\exp(\delta t\, H(q,p,t)^*)\, \Gamma[S'(q;t)] \\ &\quad = \exp(-\epsilon\, G(q,p,t+\delta t)^*)\, \Gamma[S(q;t+\delta t)] \\ &\quad = \Gamma[S'(q;t+\delta t)]. \end{aligned} \tag{27}$$

Thus we have recovered the result that S' obeys the H-J equation if S does.

A well known feature of classical (and quantum) mechanics is the dual role played by every dynamical variable. On the one hand it assumes a value (expectation value) in a given state of motion, which value does not change with time if the variable is a constant of motion. On the other hand, the variable acts as the generator of canonical (unitary) transformations. These two roles come across particularly clearly in the above analysis showing how a constant of motion acts on a solution of the H-J equation to produce another solution. Each solution $S(q;t)$ of the H-J equation describes a family of solutions of the equations of motion corresponding to all initial phases lying on $\Gamma[s]$ where s is the value of S at $t = 0$. A constant of motion $G(q,p,t)$ has a time-independent value on each one of the trajectories in the family but the value of G will generally vary over the family. This variation is present when the restriction of $G(q,p,0)$ to $\Gamma[s]$ is a non constant function of q: and exactly in that case the new solution S' to the H-J equation obtained from S through eq. (20) differs from S and describes a different family. But if $(G(q,p,0))_s$ is independent of q, then the value of G is constant over the family at $t = 0$ and so remains constant for all t. Formally we see from eqs (21b) and (17) that, the time arguments being t throughout,

$$\frac{\partial (G)_S}{\partial t} = -\left(\frac{\partial H}{\partial p_r}\right)_S \frac{\partial (G)_S}{\partial q_r}, \tag{28}$$

so that as is to be expected

$$(G(q,p,0))_s = \text{constant} = g \Longrightarrow (G(q,p,t))_S = g. \tag{29}$$

For such a solution S of the H-J equation the new solution S' is essentially the same as S, meaning that the transformation generated by G has no effect on S at all.

For the sake of complete generality we have treated the situation where both the Hamiltonian and the constant of motion were permitted to have explicit time dependences. The simpler statements valid if these dependences are absent are easily obtained from the results given already.

3. The time-independent H-J equation

We now consider conservative dynamical systems and the role of the time independent H-J equation. The general method by which solutions to this equation arise is explained in the appendix. We are particularly interested in the relationship between solutions of the time dependent and independent H-J equations and the points of similarity to the situation in quantum mechanics. We recall the latter situation first. Suppose $|\psi(t)\rangle$ is a general solution of the Schrödinger eq. (19) with a time-independent Hamiltonian operator $\hat{H}$, determined by the initial state $|\psi\rangle$ at $t = 0$:

$$|\psi(t)\rangle = \exp(-it\hat{H})\,|\psi\rangle. \tag{30}$$

This solution can be expanded linearly in terms of energy eigenstates with harmonic time dependence, say:

$$|\psi(t)\rangle = \int dE\, \phi(E)\, \exp(-iEt). \tag{31}$$

For simplicity we have assumed that $\hat{H}$ has a continuous spectrum, and $|\phi(E)\rangle$ is a (not necessarily normalised) eigenvector of $\hat{H}$ with eigenvalue E. The values of E that occur, and the corresponding vectors $|\phi(E)\rangle$, are all characteristic of the state at time $t = 0$:

$$|\psi\rangle = \int dE\, |\phi(E)\rangle. \tag{32}$$

The rule by which each of the 'pieces' $|\phi(E)\rangle$ present in $|\psi\rangle$ can be projected out of $|\psi\rangle$ reads:

$$|\phi(E)\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \exp(iEt) \exp(-it\hat{H})\,|\psi\rangle = \delta(E-\hat{H})\,|\psi\rangle. \tag{33}$$

In words: One subjects $|\psi\rangle$ to time translation by all possible amounts, multiplies by the factor $\exp(iEt)$, and integrates over all time. By this method one produces, from any given $|\psi\rangle$, energy eigenstates for all those values of E that are 'present' in $|\psi\rangle$.

There is an analogous procedure in the classical H-J theory (see, for instance, Sommerfeld [15]). Let $S(q;t)$ be a nonlinearly time dependent solution of the H-J equation

$$(H(q,p))_S + \frac{\partial S(q;t)}{\partial t} = 0. \tag{34}$$

If we introduce an energy parameter E by

$$(H(q,p))_S = E, \tag{35}$$

we regard E as a function of q and t, and conversely by inversion t as a function of q and E:

$$t = \alpha(q; E). \tag{36}$$

We now eliminate the variable t in favour of E by a Legendre transformation and define a function $W(q; E)$ by

$$\begin{aligned} W(q; E) &= S(q; t) + Et \\ &= S(q; \alpha(q; E)) + E\,\alpha(q; E). \end{aligned} \tag{37}$$

One sees then using eq. (34) that

$$\frac{\partial W(q; E)}{\partial q_r} = \left(\frac{\partial S(q; t)}{\partial q_r}\right)_{t=\alpha}. \tag{38}$$

Of course the differentiation of W is at constant E, while that of S is at constant t followed by substitution for t. Equations (35) and (38) now show that $W(q; E)$ is a solution of the time independent H-J equation:

$$(H(q; p))_W = E \tag{39}$$

Analysing this construction more closely we see the following. We know that a solution $S(q; t)$ of eq. (34) arises uniquely from an arbitrarily specified function $s(q)$ on configuration space to which S must reduce at $t = 0$. On the other hand, the solution $W(q; E)$ to eq. (39) is determined uniquely given S and E. It follows that every function $s(q)$ on configuration space must lead in a definite manner to definite solutions $W(q; E)$ of eq. (39) for each E in a certain range. It is natural to trace the emergence of $W(q; E)$ directly from $s(q)$, analogously to the quantum result (33). It is possible to do this using the geometrical constructs of section 1 and the results given in the appendix. Given $s(q)$ we first set up the n-dimensional hypersurface $\Gamma[s]$. As q runs over all of configuration space the restriction $(H)_s$ of the Hamiltonian to $\Gamma[s]$ takes up a certain range of values. For any E in this range we define an $(n-1)$-dimensional region in configuration space by the equation

$$(H(q, p))_s = E. \tag{40}$$

Suppose u_α, $\alpha = 1, 2, \ldots, n-1$, give a parametrisation of this region, so that functions $q_r(u)$ solve eq. (40). The restriction of the q_r to this region of configuration space gives a sub surface in $\Gamma[s]$ as well, of dimension $(n-1)$. The values of the momenta in this subsurface could be expressed as $\partial s(q)/\partial q_r$ with the q_r being restricted *after* the partial differentiations are done. Alternatively we can observe the restriction (40) on the q_r from the start by saying that the momenta are determined in the region considered by the equations

$$p_r(u)\,\frac{\partial q_r(u)}{\partial u_\alpha} = \frac{\partial s'(u)}{\partial u_\alpha} \tag{41a}$$

$$H(q(u), p(u)) = E, \tag{41b}$$

where

$$s'(u) = s(q(u)). \tag{42}$$

(For ease in writing the additional argument E has been omitted in both $q(u)$ and $p(u)$). Equation (41) are of just the form of eq. (A.16), so from $s(q)$ and E we have produced an $(n-1)$ parameter family of initial phases $q_r(u)$, $p_r(u)$. From the appendix we know that these are just the ingredients needed for the construction of a solution of the time independent H-J equation with energy E: we must apply all possible time translations to this family of initial phases, thereby generating an n-dimensional hypersurface Γ_E in phase space which is guaranteed to be $\Gamma[W(q; E)]$ for some $W(q; E)$ obeying eq. (39).

To explicitly find $W(q; E)$ we must use q_r as parameters over Γ_E and express the momenta p_r at points on Γ_E in the form

$$p_r = \frac{\partial W(q; E)}{\partial q_r}. \tag{43}$$

Figure 1 clarifies the construction: The n-dimensional hypersurface $\Gamma[s]$ appears as a 'line'; its $(n-1)$-dimensional subsurface defined by eq. (40) appears as a 'point'; the application of all possible time translations to this 'point' generates the (n-dimensional) 'line' Γ_E. A point on Γ_E with configuration q determines a corresponding momentum p. This phase space point (q,p) lies on the hypersurface $\Gamma[S(q;t)]$, obtained by canonical transformation from $\Gamma[s]$, for some t. The figure shows that t is determined in terms of q and E just by eqs (35) and (36)—thus clarifying the meaning of turning time into a function of q and E. If the q_r are viewed as parametrising $\Gamma[S(q; t)]$, p_r are given by

$$p_r = \frac{\partial S(q; t)}{\partial q_r}. \tag{44}$$

This is 'differentiation at constant t'. We want to express the same p_r in the manner of eq. (43) which is 'differentiation at constant E' corresponding to the use of the

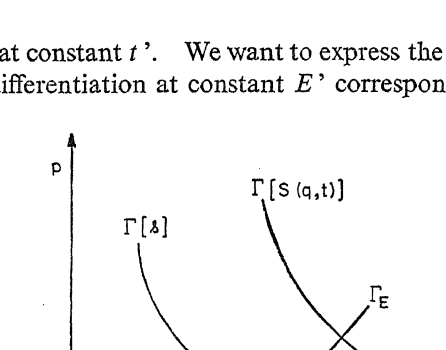

Figure 1. Generation of solutions to time-independent H-J equation.

q_r as parameters for Γ_E. This means that we must use eq. (35) to eliminate t in favour of E, and the Legendre transformation (37) gives just the correct W in terms of which we secure eq. (43).

Thus the generation of solutions $W(q; E)$ to the time-independent H-J equation starting from any configuration space function $s(q)$ is understood and seen to be similar in several respects to the situation in quantum mechanics.

4. H-J equations on a Lie group

Though it is not directly related to the topics discussed in the preceding sections, we describe very briefly in this section an extension of the H-J theory in which the single time parameter is replaced by an element of a Lie group. This is natural to consider for those dynamical systems which possess an invariance group larger than the group of time translations.

Let α^j, $j = 1, 2, \ldots, N$, be a coordinate system for an N-parameter Lie group $\mathcal{G}$, normalised so that $\alpha^j = 0$ at the identity. Let the product of the elements α, β have coordinates $f^j(\alpha, \beta)$. The functions defined by

$$\eta_j{}^k(\alpha) = \left(\frac{\partial f^k(\beta, \alpha)}{\partial \beta^j}\right)_{\beta=0} \tag{45}$$

are known to form a nonsingular matrix reducing to the unit matrix at $\alpha = 0$; and obey

$$\left[\eta_j{}^u(\alpha)\frac{\partial}{\partial \alpha^u},\ \eta_k{}^v(\alpha)\frac{\partial}{\partial \alpha^v}\right] = -C_{jk}{}^l\,\eta_l{}^w(\alpha)\frac{\partial}{\partial \alpha^w}, \tag{46}$$

where the C's are the structure constants of $\mathcal{G}$. (For a concise account see, for instance, Sudarshan and Mukunda [16] Ch. 13).

A Hamiltonian dynamical system with invariance group $\mathcal{G}$ supplies us with a set of phase space functions $G_j(q, p)$ giving a PB realisation of the Lie algebra of $\mathcal{G}$ (Saletan and Cromer [12], Ch. IX; Sudarshan and Mukunda [16] Ch. 14):

$$\{G_j(q, p), G_k(q, p)\} = C_{jk}{}^l\,G_l(q, p). \tag{47}$$

For simplicity the possible presence of neutral elements in the realisation is ignored. These relations ensure that the finite canonical transformations generated by the $G_j(q, p)$ give a realisation of $\mathcal{G}$. If $\mathcal{G}$ consisted of time translations alone, we would have just one G, the Hamiltonian.

In this framework one sees that the single H-J equation of ordinary dynamics gets enlarged to a system of H-J equations, one for each α^j, in this form:

$$\eta_j{}^k(\alpha)\frac{\partial S(q;\alpha)}{\partial \alpha^k} + (G_j(q, p))_S = 0. \tag{48}$$

The unknown here is a configuration space function S that depends also on the N parameters α^j. Equations (46) and (47) jointly ensure the integrability of this system

of H-J equations, provided we use formula (6) as well. We do not analyse this system further here, but point it out as an interesting extension of the usual theory.

5. Conclusion

We have provided a general analysis of the Hamilton-Jacobi formulation of classical dynamics laying particular emphasis on the description of symmetries and invariance. We have tried to see to what extent the classical transformation theory adapts itself naturally to this form of dynamics. This study has been motivated by certain well-known features of the Schrödinger form of quantum mechanics as well as the desire to understand better the grouping together of classical states of motion into families that is characteristic of the Hamilton-Jacobi method (cf. the remarks of Dirac [4]). Our interest has been to trace the connections existing between different elements of the theory and not to provide new techniques for solving known equations.

As an example of the better insight provided by this work into standard applications of the H-J theory, we mention the following. The cases where the H-J equation is explicitly solved are rather few in number, usually corresponding to situations where the method of separation of variables works. It is well known that these are the cases possessing specific symmetry properties; these symmetries lead to special coordinate systems in which the H-J equation separates. From sections 1 and 2 the significance of these cases is realised clearly and in a deeper way. These are just the situations wherein a maximal number of global constants of motion exist, with the property that any two of them are in involution, i.e., have vanishing PB. Precisely then does there exist a single function S that simultaneously satisfies the H-J equations corresponding to each of these constants of motion. In fact S is (essentially) uniquely determined by these equations. This is exactly like finding simultaneous eigenvectors for a complete commuting set of operators in quantum mechanics.

The discussion in this paper has been restricted (implicitly at least!) to local properties in phase space; it is extremely hard to make statements about properties in the large. In spite of this we feel that the methods used here are interesting in their own right and may suggest new points of view in analysing the semiclassical regime of quantum mechanics and its relation to classical mechanics. We hope to have successfully brought out some of the beautiful aspects of the Hamilton-Jacobi form of dynamics so that it may be viewed as something more than merely a means to solving Hamilton's equations of motion.

Appendix

The material in this appendix is presented solely for the convenience of the reader, no specifically new results being involved. Several well-known ideas have been put together in a compact form especially suitable for the applications made in the body of the paper.

Let q_r, p_r, $r=1, 2, \ldots, n$, be a canonical coordinate system for a $2n$-dimensional phase space. There are many equivalent ways of defining a canonical transformation, of which we choose the one based on the differential expression $p_r dq_r$. An

invertible one-to-one mapping of the phase space onto itself, taking a general point q_r, p_r to the point $Q_r(q, p)$, $P_r(q, p)$, is a canonical transformation if the expression

$$P_r dQ_r - p_r dq_r$$

is a perfect differential, i.e., if there exists a phase space function $w(p, q)$ such that

$$P_r dQ_r - p_r dq_r = dw(q, p). \tag{A1}$$

Here, dq_r and dp_r may be pictured as independent small changes in the coordinates of the point q_r, p_r, leading via the mapping to definite small changes dQ_r, dP_r in the coordinates of the image point Q_r, P_r. If the canonical transformation depends on one or more parameters, these will enter as arguments of w as well.

A given function $A(q, p)$ on phase space can be used as the (infinitesimal) generator of a one-parameter family of canonical transformations in a well-known way. If ϵ is the (real) parameter of the transformation, the equations of the transformation can be formally written using PB notation as:

$$Q_r(q, p, \epsilon) = q_r + \epsilon\{A, q_r\} + \frac{\epsilon^2}{2!}\{A, \{A, q_r\}\} + \cdots,$$

$$P_r(q, p, \epsilon) = p_r + \epsilon\{A, p_r\} + \frac{\epsilon^2}{2!}\{A, \{A, p_r\}\} + \cdots. \tag{A2}$$

Naturally, only canonical transformations continuously connected to the identity transformation can be produced in this way with the help of an infinitesimal generator. Putting eq. (A2) into eq. (A1), the function $w(q, p, \epsilon)$ can be calculated as a power series in ϵ. Up to second order, the result is

$$P_r dQ_r - p_r dq_r = d\left[\epsilon(A - p_r\{q_r, A\}) - \frac{\epsilon^2}{2}\{A, p_r\{q_r, A\}\}\cdots\right]. \tag{A3}$$

While q_r and p_r appear as independent variables in eq. (A 1), one can obtain useful results by restricting them in various ways. Let M be an m-dimensional hypersurface embedded in the $2n$-dimensional phase space, and let u_α, $\alpha = 1, 2, \ldots, m$, give a real parametrisation of M in the sense that the coordinates of points of M can be written as $q_r(u)$, $p_r(u)$ with the u_α being independently variable. The canonical transformation applied to the points of M leads to another m-dimensional hypersurface, M' say, with the image of the point $q_r(u)$, $p_r(u)$ on M being the point $Q_r(u)$, $P_r(u)$ on M'; thus the u_α serve as parameters for M' as well. We now restrict dq_r, dp_r to be just those possible small changes in q_r, p_r that can be produced by independent small changes in the u_α, so that along with q_r, p_r, the point $q_r + dq_r$, $p_r + dp_r$ also belongs to M. The canonical nature of the transformation then leads to an equation of the form

$$P_r(u)\frac{\partial Q_r(u)}{\partial u_\alpha} - p_r(u)\frac{\partial q_r(u)}{\partial u_\alpha} = \frac{\partial w(u)}{\partial u_\alpha}. \tag{A4}$$

From here we conclude that

$$\frac{\partial}{\partial u_\beta}\left(P_r \frac{\partial Q_r}{\partial u_\alpha}\right) - \frac{\partial}{\partial u_\alpha}\left(P_r \frac{\partial Q_r}{\partial u_\beta}\right) = \frac{\partial}{\partial u_\beta}\left(p_r \frac{\partial q_r}{\partial u_\alpha}\right) - \frac{\partial}{\partial u_\alpha}\left(p_r \frac{\partial q_r}{\partial u_\beta}\right), \tag{A5}$$

which is in fact an explicit expression of the invariance of the LB's among the u_α under a canonical transformation:

$$[u_\alpha, u_\beta]_{Q,P} = [u_\alpha, u_\beta]_{q,p}. \tag{A6}$$

In the remainder of this appendix we summarise briefly how general solutions to the time dependent H-J equation for a general Hamiltonian $H(q, p, t)$ and to the time independent H-J equation for a conservative Hamiltonian $H(q, p)$, can in principle be constructed (For detailed discussions see Caratheodory [3], Rund [11]). The key idea to be used is of course the fact that time evolution according to Hamilton's equations of motion,

$$\dot{Q}_r = \{Q_r, H(Q, P, t)\}, \dot{P}_r = \{P_r, H(Q, P, t)\} \tag{A7}$$

is the continuous unfolding of a canonical transformation. As preparation for both cases we consider first the following situation. Choose as above an m-dimensional hypersurface M with parameters u_α and consider the m-fold infinity of phases $q_r(u)$, $p_r(u)$ at an initial time $t=0$. Each of these initial phases gives rise via eq. (A7) to a corresponding phase space trajectory, the phase at time t on a general one of these trajectories being $Q_r(t; u)$, $P_r(t; u)$. Equation (A6) now states that the LB's $[u_\alpha, u_\beta]$ are independent of time. It follows that these LB's vanish for all time over the entire family of trajectories if they vanish everywhere on M, i.e., the hypersurface M is such that for a suitable function $s(u)$ we have

$$p_r(u) \frac{\partial q_r(u)}{\partial u_\alpha} = \frac{\partial s(u)}{\partial u_\alpha}. \tag{A8}$$

Let these equations be fulfilled. Then we see that, for the m-parameter family of trajectories being considered, eq. (A4) guarantees the existence of a function $w(t; u)$ such that

$$P_r(t; u) \frac{\partial Q_r(t; u)}{\partial u_\alpha} = \frac{\partial}{\partial u_\alpha}(s(u) + w(t; u)). \tag{A9}$$

In this general discussion all values of $m \leqslant n$ are permitted. In particular if $m<n$, knowledge of $s(u)$ and $q_r(u)$ does not suffice to determine $p_r(u)$ completely via eq. (A8).

To arrive at general solutions of the time dependent H-J equation for a general Hamiltonian $H(Q, P, t)$, we specialise the above discussion by taking M to be of dimension n, $m=n$, and assuming that the q_r can be chosen as the independent parameters u_α on M. Then eq. (A8) does determine the momenta p_r on M as functions of the q_r,

$$p_r = \frac{\partial s(q)}{\partial q_r}, \tag{A10}$$

and M becomes a hypersurface $\Gamma[s]$ of the type introduced in section 1, determined completely by a given function $s(q)$ on configuration space. The function $s(q)$ then produces, according to the previous paragraph, an n-parameter family of trajectories with the phase at time t on a general member being $Q_r(t; q)$, $P_r(t; q)$. We assume next for each t that the Q_r are independent functions of the q_r. In that case, eq. (A9) shows that there exists a function $S(Q; t)$ such that at each time t we have

$$P_r\, dQ_r = dS(Q; t). \tag{A11}$$

The family of trajectories is therefore such that the momenta are always determined by the positions through

$$P_r = \frac{\partial S(Q; t)}{\partial Q_r}. \tag{A12}$$

But the family has been obtained by integrating the equations of motion (A7) for a particular collection of initial phases, so eq. (A12) must be consistent with eq. (A7). This means, on taking the total time derivative of both sides of eq. (A12) and then using eqs (A 7) and (A 12) in the result,

$$\left(-\frac{\partial H(Q, P, t)}{\partial Q_r}\right)_S = \frac{\partial^2 S}{\partial Q_r \partial t} + \frac{\partial^2 S}{\partial Q_r \partial Q_s}\left(\frac{\partial H(Q, P, t)}{\partial P_s}\right)_S,$$

i.e.,

$$\frac{\partial}{\partial Q_r}\left[(H(Q, P, t))_S + \frac{\partial S}{\partial t}\right] = 0. \tag{A13}$$

Since eq.(A11) leaves S undetermined up to the addition of a time-dependent constant, we see that S may be taken to obey the H-J equation

$$(H(Q, P, t))_S + \frac{\partial S}{\partial t} = 0. \tag{A14}$$

This still leaves S undetermined up to the addition of a time-independent constant. This ambiguity is eliminated by comparing eq. (A11) with eq. (A10) and requiring that at $t=0$ we have

$$S(Q; 0) = s(Q). \tag{A15}$$

In this way, each given function $s(Q)$ on configuration space leads to a corresponding uniquely determined solution of the time dependent H-J eq. (A14).

Finally suppose $H(Q,P)$ is a Hamiltonian with no explicit time dependence and we wish to build up solu ions to the corresponding time-independent H-J equation. In the general discussion preceding eq. (A9) we take $m=n-1$ and choose the functions $q_r(u)$ in some way. Thus the initial configurations belong to an $(n-1)$-dimensional

region of configuration space. To determine the initial momenta $p_r(u)$ we impose n equations

$$p_r(u)\frac{\partial q_r(u)}{\partial u_\alpha} = \frac{\partial s'(u)}{\partial u_\alpha}, \tag{A16a}$$

$$H(q(u), p(u)) = E. \tag{A 16b}$$

Here $s'(u)$ is some function of the $(n-1)$ parameters u_α and E an arbitrarily chosen energy. (We assume these are independent equations for the $p_r(u)$. and skip writing the more correct form $p(u; E)$). This $(n-1)$ parameter family of initial phases produces an $(n-1)$ parameter family of trajectories filling out an n-dimensional region in phase space. The phase at time t on one of these trajectories can be written $Q_r(t; u; E)$, $P_r(t; u; E)$ and eq. (A9) guarantees the existence of a $V(t; u; E)$ such that

$$P_r(t; u; E)\frac{\partial Q_r(t; u; E)}{\partial u_\alpha} = \frac{\partial V(t; u; E)}{\partial u_\alpha}. \tag{A17}$$

Energy conservation and eq (A 16b) jointly give

$$H(Q(t; u; E), P(t; u; E)) = E. \tag{A18}$$

Now we must extend eq. (A17) to cover the partial derivatives with respect to t also. For this we show first that since H has no explicit time dependence the LB's $[t, u_\alpha]$ are conserved:

$$\begin{aligned}\frac{\partial}{\partial t}([t, u_\alpha]_{Q,P}) &= \frac{\partial}{\partial t}\left(\frac{\partial Q_r}{\partial t}\frac{\partial P_r}{\partial u_\alpha} - \frac{\partial Q_r}{\partial u_\alpha}\frac{\partial P_r}{\partial t}\right)\\ &= \frac{\partial}{\partial t}\left(\frac{\partial H(Q,P)}{\partial P_r}\frac{\partial P_r}{\partial u_\alpha} + \frac{\partial H(Q,P)}{\partial Q_r}\frac{\partial Q_r}{\partial u_\alpha}\right)\\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial u_\alpha}H(Q,P) = \frac{\partial}{\partial u_\alpha}\frac{\partial}{\partial t}H(q(u), p(u)) = 0.\end{aligned} \tag{A19}$$

[The arguments t, u, E of Q, P were omitted for simplicity]. Here we used only the conservation of H but not the further property that for all members of the family of trajectories considered H has the same value E. If we do take account of this property then the same steps as in eq. (A 19) but with $\partial/\partial t$ omitted show that the LB's $[t, u_\alpha]$ in fact have the constant values zero:

$$[t, u_\alpha] = \frac{\partial}{\partial u_\alpha}H(q(u), p(u)) = 0. \tag{A20}$$

We express this as

$$\frac{\partial}{\partial u_\alpha}\left(P_r\frac{\partial Q_r}{\partial t}\right) = \frac{\partial}{\partial t}\left(P_r\frac{\partial Q_r}{\partial u_\alpha}\right), \tag{A21}$$

recall that $V(t; u; E)$ in eq. (A 17) is at this stage undetermined up to addition of a function of t (and E) alone, and so conclude that V may be chosen so that we can write

$$P_r dQ_r = dV(t; u; E) \tag{A22}$$

with all n quantities t, u_r capable of independent variation. As the last step we assume that $Q_r(t; u; E)$ are n independent functions of t, u_α; this means that the Q_r can serve as independent parameters for the n-dimensional region in phase space filled out by the trajectories. If then $V(t; u; E)$ is written as $W(Q; E)$ eq. (A 22) gives

$$P_r = \frac{\partial W(Q; E)}{\partial Q_r}, \tag{A23}$$

and this coupled with eq.(A18) shows that W is a solution of the time-independent H-J equation

$$H\left(Q, \frac{\partial W}{\partial Q}\right) = E. \tag{A24}$$

To summarise, each solution of this equation arises by choosing an $(n-1)$ parameter family of initial positions $q_r(u)$, a function $s'(u)$, and an energy E in such a way that the initial momenta can be obtained from eq. (A 16), and such that in the resulting family of trajectories Q_r can replace t, u_α as independent variables. The n-dimensional region of phase space filled out by the trajectories determined by this $(n-1)$ parameter family of initial phases forms the hypersurface $\Gamma[W(q; E)]$. The construction just outlined is essentially the one given by Einstein [5], while the one described by Schrödinger [13] is a special case of this one.

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Subsonic and supersonic disturbances in the generation of surface waves in a liquid layer adjacent to a high-speed gas-stream

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Abstract. In an effort to shed further light upon the nature of " supersonic " disturbances as distinct from that of ' subsonic ' disturbances in parallel compressible flows, this paper makes an investigation of the stability characteristics of the surface waves generated in a liquid layer adjacent to a high-speed gas-stream. It turns out that the nature of the surface waves generated in the liquid layer depends markedly upon the type of disturbances present in the high-speed gas-stream. For the case of ' subsonic ' disturbances it is shown that the energy transfer from the gas stream to the surface waves is contributed predominantly by the Fourier component of the normal gas-pressure force-field in phase with the slope of the wavy surface. For the case of ' supersonic ' disturbances, this energy transfer is shown to be predominantly due to the component of the pressure-field in phase with the surface-wave displacement and is related to the presence of travelling periodic waves in the gas-stream—this energy transfer is shown to promote always the growth or the surface waves.

Keywords. Subsonic disturbance;supersonic disturbance; surface waves.

1. Introduction

Contemporary literature shows that the precise role of what are defined as the ' supersonic ' disturbances in the context of the stability of parallel compressible flows is not entirely clear, while they are supposed sometimes, as Miles [5] from the study of a perturbed vortex sheet in a compressible fluid found, to impose further restrictions upon the sufficient conditions for stability. The present author [6] carried out a phase-plane analysis of the linearised inviscid pressure disturbances in parallel compressible flows and found that while a given basic flow executes a unique set of ' subsonic ' oscillations, no uniqueness exist in executing supersonic oscillations. In an effort to shed further light upon the nature of supersonic disturbances, this paper makes an investigation of the stability characteristics of the surface waves generated in a liquid layer adjacent to a high-speed gas-stream—a problem which is of interest, for instance, in the flight of reentry vehicles in earth's atmosphere. In this direction, we consider a simple wave train that is stationary relative to the surface waves in the liquid layer superposed on the incoming high-speed gas-stream, and then investigate the conditions under which the reaction of the disturbed flow is just sufficient to sustain the surface waves. This, in particular, involves the prediction of the energy transfer from the high-speed gas-stream to the surface waves. The liquid is assumed to be incompressible and inviscid, and the gas to be compressible and inviscid. A justification of this inviscid model for the determination of a first approxi-

mation to the disturbed motion of the high-speed gas-stream and the consequent energy transfer to the surface waves in the liquid layer comes from the fact that the predominant force field at the interface is due to the pressure field of the high-speed gas-stream which is not affected by the presence of the viscous boundary layer and which is considerably larger than the viscous frictional force at the interface. In the following we shall avoid the question of formulating the mechanism by which the surface-wave energy is dissipated.

It turns out that the nature of the surface waves generated in the liquid layer depends significantly on the type of disturbances present in the high-speed gas-stream. For the case of subsonic disturbances, it is shown that the energy transfer from the gas stream to the surface waves is contributed predominantly by the Fourier component of the normal gas-pressure force field in phase with the slope of the wavy surface. (This appears to be similar to the so-called " sheltering " mechanism proposed by Jefreys [1, 2] in the context of the generation of surface waves in water by a wind blowing over it, which is based on a flow-separation from leeward-side of each wave on the surface so that there is a pressure difference between the windward- and leeward side of each crest, and if the wind speed exceeds the wave speed, energy is thereby transferred from the air to the water). In calculating this energy transfer, use is made of the fact that this energy transfer is primarily determined by the conditions at the critical point where, $u(y)=c$, $u(y)$ being the gas speed, since it is known that the flexure of the interface causes the wave train to travel in the flow direction with a speed c, and the critical point may then occur away from the interface—an important discovery due to Miles [4] in the context of the generation of surface waves in water by a wind blowing over it. For the case of supersonic disturbances, this energy transfer is shown to be predominantly due to the component of the pressure-field in phase with the surface-wave displacement and is related to the presence of travelling periodic waves in the gas stream—this energy transfer is shown to promote always the growth of the surface waves.

2. Surface-wave generation by subsonic disturbances

Consider an initially stationary liquid of infinite depth in the region $y<0$, and a high-speed gas-stream in the x-direction past the liquid surface through the region $y>0$. The high-speed gas-stream is taken to be parallel, i.e., the flow properties are considered to vary negligibly over distances comparable with the wavelength, and in the direction of the surface waves in the liquid layer so that the disturbances due to the wavy interface given by

$$y_0 = a \exp [i\alpha (x-ct)] \tag{1}$$

are two-dimensional and strictly periodic in the x-coordinate. In the linearised theory for the disturbances that we develop here, one ignores the flow-separation from the wave-crests and requires the surface wave-amplitude to be small in comparison with the wavelength and also that the surface curvature be sufficiently small. In the present case, the motion of the liquid below the interface is neglected.

Then the perturbations in the flow-properties—longitudinal—and transverse-velocity components u and v, pressure p, density ρ, and temperature T are considered to

be of the form $\exp[i\alpha(x-ct)]$ times an amplitude function of y, where c is the complex wave velocity $c=c_r+ic_i$, t being the time. Under the approximations of parallel flows, the equations of continuity, motion, energy, and state yield on reduction five linear differential equations for the amplitudes of the linearised disturbances (see Lees and Lin [3]):

$$\rho\,(u-c)\,\gamma'+\rho\,(\phi'+if)+\rho'\phi=0 \tag{2}$$

$$\rho\,[i\,(u-c)\,f+u'\phi]=-\frac{i\pi}{\gamma M_1^2} \tag{3}$$

$$\alpha^2\rho\,[i\,(u-c)\,\phi]=-\frac{\pi'}{\gamma M_1^2} \tag{4}$$

$$\rho\,[i\,(u-c)\,\theta+T\phi']=-\,(\gamma-1)\,\rho T\,(\phi'+if) \tag{5}$$

$$\frac{\pi}{p}=\frac{\gamma'}{\rho}+\frac{\theta}{T} \tag{6}$$

where we have considered a perfect gas with constant ratio of specific heats γ. The disturbances are defined by

$$\hat{u},\ \hat{v},\ \hat{p},\ \hat{\rho},\ \hat{T}=(f,\ \alpha\phi,\ \pi,\ \gamma',\ \theta)\exp[i\alpha\,(x-ct)] \tag{7}$$

and the nondimensional parameters are

$$\gamma=\frac{C_p}{C_v},\quad M_1^2=\frac{V_\infty^2}{\gamma RT_\infty} \tag{8}$$

where the subscript ∞ denotes the conditions on the gas stream at infinity, R is the gas constant, and the primes denote differentiation with respect to y.

The linearised boundary condition at the interface is

$$y=0:\phi=0 \tag{9a}$$

and the conditions of boundedness at infinity (valid for subsonic disturbances) are

$$y\to\infty:f,\ \phi,\ \pi,\ \gamma',\ \theta\to 0 \tag{9b}$$

One deduces from eqs (2)–(6),

$$\frac{d}{dy}\left[\frac{(u-c)\phi'-u'\phi}{T-M_1^2\,(u-c)^2}\right]-\frac{\alpha^2(u-c)}{T}\phi=0. \tag{10}$$

The disturbances in the gas stream are then classified as subsonic, sonic, or supersonic according as the phase velocity in the direction of the gas stream is, in magnitude, below, equal to, or above the local speed of sound, i.e.,

$$\begin{aligned} &\text{subsonic disturbances} &&: && M_1^2\,(u-c)^2<T \\ &\text{sonic disturbances} &&: && M_1^2\,(u-c)^2=T \\ &\text{supersonic disturbances} &&: && M_1^2\,(u-c)^2>T \end{aligned} \tag{11}$$

From eq. (4), one finds

$$\pi(0) = -\int_0^\infty \pi' dy = \gamma M_1^2 \int_0^\infty \frac{i\alpha^2(u-c)}{T}\,\phi\, dy \tag{12}$$

which implies that the pressure distribution at the interface is governed by the cumulative action of the disturbance over the whole flow field, and is affected very little by the phenomena near the interface. Using eq. (10), one obtains

$$p_s = \pi(0) = -\frac{i\gamma M_1^2(u-c)}{T-M_1^2(u-c)^2}\phi'. \tag{13}$$

Since the work done by the pressure forces on the wavy interface is

$$c\int p_s \frac{\partial y_0}{\partial x}\,dx$$

it follows that in the absence of viscosity, the rate of energy transfer from the gas stream to unit area of the wavy interface is proportional to $-\mathrm{Im}\{p_s\}$. The growth of the surface waves in the liquid layer is then supposed to occur when the energy supply exceeds the rate of energy dissipation by viscosity in the fluid below the interface.

Note from eq. (13)

$$-\mathrm{Im}\{p_s\} \simeq \frac{\gamma M_1^2 c_i}{T-M_1^2(u-c)^2}\,\mathrm{Im}\{\varphi^1\} \tag{14}$$

Next multiply eq. (10) by ϕ^*, the complex conjugate of ϕ, subtract from the resulting equation its complex conjugate, and then there follows

$$\frac{d}{dy}(\phi'\phi^*-\phi^{*\prime}\phi) - \left[\frac{T'-2M_1^2(u-c)u'}{T-M_1^2\,(u-c)^2}\right]\phi'\phi^* + \left[\frac{T'-2M_1^2(u-c)u'}{T-M_1^2(u-c)^2}\right]\phi^{*\prime}\phi$$

$$+ (c-c^*)\left[\frac{\dfrac{T'-2M_1^2(u-c)u'}{T-M_1^2(u-c)^2}\,u'-u''}{|u-c|^2}\right]|\phi|^2 = 0 \tag{15}$$

where we have considered only slightly amplified surface waves.

Introducing the 'Wronskian',

$$W = \frac{i}{2}\frac{(\phi\phi^{*\prime}-\phi'\phi^*)}{T-M_1^2(u-c)^2} \tag{16}$$

eq. (15) becomes

$$\frac{dW}{dy} - c_i \left[\frac{\left\{ \dfrac{u'}{T - M_1^2 (u-c)^2} \right\}'}{|u-c|^2} \right] |\phi|^2 = 0. \tag{17}$$

Letting $c_i \to 0$, one obtains

$$W = \text{const}$$

except possibly at the critical point y_c, which now lies in the real interval $(0, \infty)$, and W has a discontinuity at that point. Integrate eq. (17) from y_c^- to y_c^+, then

$$W(y_c^+) - W(y_c^-) = [W]_c$$

$$= \int_{u_c^-}^{u_c^+} \frac{\left\{ \dfrac{u'}{T - M_1^2 (u-c)^2} \right\}' |\phi|^2 \, c_i \, du}{u' \{(u-c_r)^2 + c_i^2\}}. \tag{18}$$

A straightforward calculation gives (see Shivamoggi [7]),

$$W(y_c^+) - W(y_c^-) = [W]_c = \pi \frac{(u'/T)'_{u=c}}{u'_c} |\phi_c|^2 \tag{19}$$

provided in this limit, $c_i > 0$, $u_c^+ > u_c^-$. Since $\phi \to 0$ for $y \to \infty$, $W \equiv 0$, for $y > y_c$, so that

$$W(y) = -[W]_c, \quad y < y_c \tag{20}$$

Noting that $\phi = [u(y) - c]$ is a solution of eq. (10) in the limit $\alpha \to 0$, one finds for the neighbourhood of the interface

$$W(y) = \frac{i}{2} \frac{(u-c)}{T - M_1^2 (u-c)^2} [\phi^{*\prime} - \phi']$$

or

$$W(y) = \frac{(u-c)}{T - M_1^2 (u-c)^2} \operatorname{Im} \{\phi'\} \tag{21}$$

Therefore from eq. (14), (19)–(21), there follows as $c_i \to 0$, that

$$-\operatorname{Im} \{p_s\} \simeq -\pi \gamma M_1^2 \frac{(u'/T)'_{u=c}}{u'_c} |\phi_c|^2 \tag{22}$$

which shows that in the absence of viscosity the energy from the high-speed gas-stream through the subsonic disturbances is positive (such as to promote the growth of the surface waves in the liquid layer) when

$$(u'/T)'_{u=c} < 0 \tag{23}$$

3. Surface-wave generation by supersonic disturbances

In this case the gas flowing past the liquid surface is assumed to have a uniform velocity U and uniform temperature T. Noting that for the case of supersonic disturbances, one has periodic waves, it follows from eq. (13) that the pressure exerted by the gas at the interface is equal to

$$\triangle p_g(0) = \left[\frac{\rho_g i\gamma M_1^2 a\alpha (U-c)}{\left\{\frac{M_1^2 (U-c)^2}{T} - 1\right\}^{1/2}}\right] \exp[i\alpha x - ct]. \tag{24}$$

The linearised equations of motion of the liquid are

$$\rho_l \frac{\partial \mathbf{q}_l}{\partial t} = -\nabla(\triangle p_l) \tag{25}$$

$$\nabla \cdot \mathbf{q}_l = 0 \tag{26}$$

where $\mathbf{q}_l$ is the liquid velocity. The boundary conditions are,

(1) condition at infinity : $|\mathbf{q}_l| \to 0$ as $y \to -\infty$ (27)

(2) kinematic interfacial condition : $y = 0$: $\partial y_0/\partial t = v_l$ (28)

(3) dynamic interfacial condition: $\triangle p_g = \triangle p_l$. (29)

The solutions to eqs (25), (26), subject to conditions (27), (28) are

$$u_l = aac \exp(\alpha y) \exp[i\alpha (x-ct)]$$

$$v_l = -iaac \exp(\alpha y) \exp[i\alpha (x-ct)]$$

$$\triangle p_l = \rho_l aac^2 \exp(\alpha y) \exp[i\alpha (x-ct)] \tag{30}$$

so that (29) gives

$$c^2 = \frac{\rho_g}{\rho_l} \frac{i\gamma M_1^2 (U-c)}{\{M_1^2 [(U-c^2)/T] - 1\}^{1/2}} \tag{31}$$

from which it is clear that for supersonic disturbances

$$\mathrm{Im}\{c\} > 0 \tag{32}$$

so that there always appears to be an energy transfer from the gas stream through the supersonic disturbances to the surface waves leading to their amplification—in significant contrast to the situation associated with the subsonic disturbances.

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Laminar compressible boundary layer flow at a three-dimensional stagnation point with vectored mass transfer

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Abstract. The effect of surface mass transfer velocities having normal, principal and transverse direction components ('vectored' suction and injection) on the steady, laminar, compressible boundary layer at a three-dimensional stagnation point has been investigated both for nodal and saddle points of attachment. The similarity solutions of the boundary layer equations were obtained numerically by the method of parametric differentiation. The principal and transverse direction surface mass transfer velocities significantly affect the skin friction (both in the principal and transverse directions) and the heat transfer. Also the inadequacy of assuming a linear viscosity-temperature relation at low-wall temperatures is shown.

Keywords. Skin friction; heat transfer; vectored mass transfer; noddle and saddle points; variable fluid properties: parametric differentiation.

List of Symbols

a, b = velocity gradients in x and y directions, respectively
c = ratio of velocity gradients, b/a
C_{f_x}, C_{f_y} = skin-friction coefficients along x and y directions, respectively
f, s = dimenionless stream functions such that $f'=u/u_e$ and $s'=v/v_e$
f_w = mass transfer parameter, $-(\rho w)_w/(\rho_e\mu_e a)^{1/2}$
g = dimensionless enthalpy, h/h_e
g_w = cooling parameter for the wall, h_w/h_e
h = enthalpy
Pr = Prandtl number
q = heat transfer rate
Re_x = local Reynolds number, $u_e x/\nu_e$
St = Stanton number
T = temperature
u, v, w = velocity components along x, y, z directions, respectively
x, y, z = principal, transverse and normal directions, respectively
η = similarity variable, $(\rho_e a/\mu_e)^{1/2} \int_0^z (\rho/\rho_e)dz$
μ = coefficient of viscosity
ν = kinematic viscosity
ρ = density
τ_x, τ_y = dimensional shear stress functions
ω = exponent in the power-law variation of viscosity

Superscript

' (prime) = differentiation with respect to η

Subscripts

e = condition at the edge of the boundary layer

w = condition at the surface $z=\eta=0$.

1. Introduction

With the advent of high-speed flight vehicles, an important aspect to be considered in the study of boundary layers are surface mass transfer effects involving normal and tangential components for two-dimensional or axisymmetric bodies and normal, principal and transverse direction components for three-dimensional bodies. These are called ' vectored ' suction or injection and finds application in many situations, such as, (1) oblating thermal protection systems of lifting re-entry vehicles, (2) in studies on the effect of velocity slip and temperature jump on flow characteristics in rarefied flows, (3) film and transpiration cooling of turbine blades, etc. For two-dimensional and axisymmetric bodies vectored mass transfer, involving only the normal component, that is, suction and/or injection has been discussed extensively in the past. In recent years, the other case has also been studied as boundary layer flow over continuous moving solid surfaces (Sakiadis [11]; Bourne and Elliston [2]; Ackroyd [1]; Williams and Johnson [15]; Libby [6]). Inger and Swean [4] and Scala and Sutton [12] have studied the effects of vectored suction and injecton (i.e., taking into account both the normal and tangential components) on laminar compressible boundary layer flow of a gas with constant properties ($\rho\mu$=constant, Pr=1 and 0·7) at a constant pressure, and at an axisymmetric stagnation point, respectively. Nath and Muthanna [8] extended the analysis of Scala and Sutton [7] to include the effects of variable gas properties ($\rho \propto T^{-1}$, $\mu \propto T^{\omega}$, Pr=0·7). In particular, they have studied the effects of vectored suction and injection on the flow and heat transfer at the stagnation point of a two-dimensional body (a cylinder) and an axisymmetric body (a sphere).

In this paper, we have investigated the effect of surface mass transfer velocities having normal, principal and transverse direction components (vectored suction and injection) on the steady, laminar compressible boundary layer flow of a gas with variable properties at a three-dimensional stagnation point both for nodal and saddle points of attachment. The governing equations were solved by the method of parametric differentiation. The foregoing problem for a stationary wall ($f'_w = s'_w = 0$) has been studied by Libby [5] assuming constant gas properties and further by Wortman *et al* [16] and Vimala and Nath [14] who employed variable gas properties. It may be remarked that the studies made by Libby [5], Wortman *et al* [16] and Vimala and Nath [14] are only a particular case of the present problem.

2. Governing equations

The governing equations in dimensionless form for steady laminar hypersonic flow of a gas with variable properties (i.e., $\rho \propto T^{-1}$, $\mu \propto T^{\omega}$, Pr=0·7) in the neighbourhood

of the stagnation point of a three-dimensional body, taking into account the effects of ' vectored ' suction or injection are (Libby [5]; Wortman *et al* [16]; Vimala and Nath [14])

$$f''' + (\omega - 1)\, g' f''\, g^{-1} + [(f + cs) f'' + g - f'^2]\, g^{1-\omega} = 0 \tag{1a}$$

$$s''' + (\omega - 1)\, g' s''\, g^{-1} + [(f + cs) s'' + c(g - s'^2)]\, g^{1-\omega} = 0 \tag{1b}$$

$$g'' + (\omega - 1)\, g'^2\, g^{-1} + Pr\,(f + cs)\, g'\, g^{1-\omega} = 0. \tag{1c}$$

The appropriate boundary conditions for vectored mass transfer are

$$f(0) = f_w,\ f'(0) = u_w/u_e = f'_w,\ f'(\infty) \to 1 \tag{2a}$$

$$s(0) = 0,\ s'(0) = v_w/v_e = s'_w,\ s'(\infty) \to 1 \tag{2b}$$

$$g(0) = g_w,\ g(\infty) \to 1. \tag{2c}$$

Here $f_w \gtrless 0$ according to whether there is suction or injection. We remark that $f_w > 0$, $f'_w > 0$, $s'_w \gtrless 0$ corresponds to downstream vectored suction, $f_w < 0$, $f'_w > 0$, $s'_w \gtrless 0$, to downstream vectored injection, $f_w > 0$, $f'_w < 0$, $s'_w \gtrless 0$ to upstream vectored suction and $f_w < 0$, $f'_w < 0$, $s'_w \gtrless 0$ to upstream vectored injection. It may be noted that $\omega = 0{\cdot}5$ corresponds to conditions encountered in hypersonic flight, $\omega = 0{\cdot}7$ corresponds to low-temperature flows and $\omega = 1$ represents the constant density viscosity product simplification which has been widely used in studying boundary layer flows (Gross and Dewey [3]). We have taken the Prandtl number to be constant, since its variation in boundary layer, for most atmospheric flight problems, is quite small (Wortman *et al* [16]). It is to be mentioned that most shapes of practical interest range from sphere ($c = 1$) to cylinder ($c = 0$) and the saddle shapes ($-1 < c < 0$) are included in the analysis for the sake of completeness (Wortman *et al* [16]).

The skin friction coefficients along x and y directions are given by (Libby[5]).

$$C_{f_x} = 2\tau_x/\rho_e u_e^2 = 2\,(Re_x)^{-1/2}\, \bar{F}''(0) \tag{3a}$$

$$C_{f_y} = 2\tau_y/\rho_e u_e^2 = 2\,(Re_x)^{-1/2}\,(v_e/u_e)\, \bar{S}''(0). \tag{3b}$$

Similarly, the heat transfer coefficient in terms of Stanton number can be expressed as (Libby [5]).

$$St = q_w/[(h_e - h_w)\, \rho_e u_e] = (Re_x)^{-1/2}\, \bar{G}'(0) \tag{4}$$

where

$$\bar{F}''(0) = g_w^{\omega-1} f''(0),\quad \bar{S}''(0) = g_w^{\omega-1} s''(0) \tag{5a}$$

$$\bar{G}'(0) = g_w^{\omega-1} g'(0)/[(Pr\,(1 - g_w)]. \tag{5b}$$

3. Results and discussion

The governing equations (1*a*–1*c*) under the boundary conditions (2*a*–2*c*) were solved by the method of parametric differentiation for various values of f_w, f'_w ($f'_w \geqslant 0$), s'_w ($s'_w \geqslant 0$), ω, c and g_w. We have not taken into account upstream vectoring because we have investigated the flow in the stagnation region. The detailed description of the method of parametric differentiation along with its application to various problems is given by Rubbert [9], Rubbert and Landahl [10]. Tan and DiBiano [13] and Na and Turski [7]; hence it is not described here. The starting solutions for carrying out computations, corresponding to $f'_w = s'_w = 0$ for various values of f_w, ω, c and g_w, were obtained from Libby [5]. Runge-Kutta-Gill method was employed for numerical integration of the differential equations, taking step size 0·05 and the edge of the boundary layer to be between 4·0 and 6·0. It was observed that a further reduction in step size affected the results only in the fourth decimal place. Convergence was considered to have been achieved when the wall parameters $f''(0)$, $s''(0)$ and $g'(0)$ in two successive iterates agreed within 10^{-4}.

Though, numerical solutions were obtained for various values of the parameters, for the sake of brevity, only some representative results are presented in the following figures. The velocity, enthalpy, shear-stress and heat-transfer profiles for a saddle point of attachment ($c = -0{\cdot}5$), with vectored suction and injection are presented in figures 1–3. The profiles corresponding to a nodal point of attachment ($c \geqslant 0$) have not been shown for lack of space.* It is observed that for given f'_w and s'_w, the effect of suction ($f_w > 0$) is to make the velocity and enthalpy profiles more steep, whatever may be the value of ω and c. The velocity and enthalpy profiles when $\omega = 0{\cdot}5$ possess a point of inflexion as is evidenced by a maximum in $f''(\eta)$, $s''(\eta)$ and $g'(\eta)$ whatever may be the value of c or f_w. When $\omega = 1$, the profiles have a point of inflexion only for injection ($f_w < 0$). The effect of downstream vectored suction ($f_w > 0$, $f'_w > 0$, $s'_w > 0$) is to increase the skin friction profiles (both in the principal and transverse di-

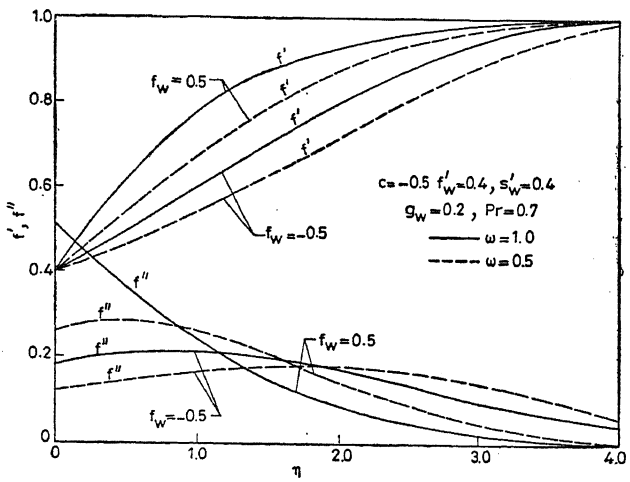

Figure 1. Variation of f' and f'' with η.

*The profiles may be obtained from the authors.

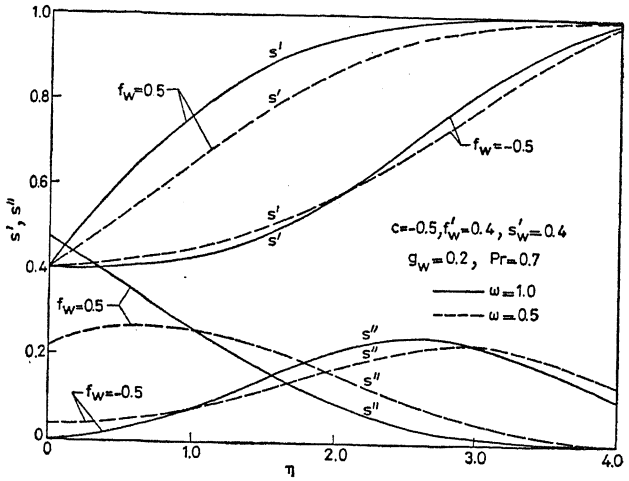

Figure 2. Variation of s' and s'' with η.

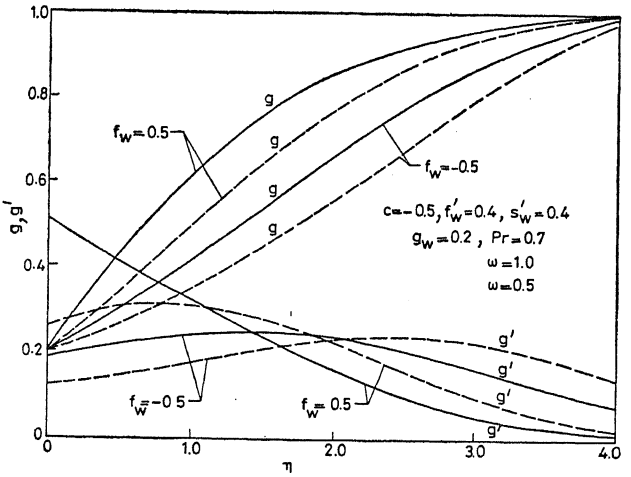

Figure 3. Variation of g and g' with η.

rections) and heat transverse profile while vectored injection ($f_w<0, f'_w>0, s'_w>0$) does the reverse. An interesting feature observed is that when $c=-0{\cdot}5$, the transverse velocity s' is influenced significantly by mass transfer. With injection, the transverse skin friction at the surface is very small and compared to suction, the difference between the $\omega=1$ and $\omega=0{\cdot}5$ profiles both for s' and s'' is comparatively less.

Figures 4–6 show the effect of f_w (the normal component of mass transfer) on the skin friction parameters $\overline{F}''(0)$ and $\overline{S}''(0)$ and heat transfer parameter $\overline{G}'(0)$ for various values of c. We see that as f_w increases $\overline{F}''(0)$ and $\overline{G}'(0)$ exhibit some interesting features. As c decreases from zero ($c<0$), the parameters $\overline{F}''(0)$ and $\overline{G}'(0)$ decrease, till a critical negative value of c is reached, after which these parameters start increasing. But this trend is not observed in the $\overline{S}''(0)$ profile. Figures 7–9 show the effect of the surface mass transfer velocity f'_w in the principal direction on the skin friction

parameters $\overline{F}''(0)$ and $\overline{S}''(0)$ and heat transfer parameter $\overline{G}'(0)$. The effect of increasing f'_w results in a decrease in $\overline{F}''(0)$ while $\overline{S}''(0)$ and $\overline{G}'(0)$ increase for fixed values of c, f_w, s'_w and g_w. The decrease in $\overline{F}''(0)$ as f'_w increases is more at higher values of

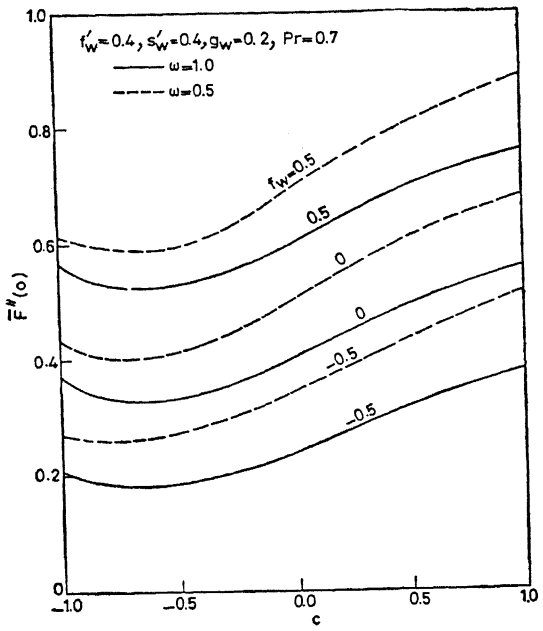

Figure 4. Effect of f_w on the variation of $\overline{F}''$ (0) with c.

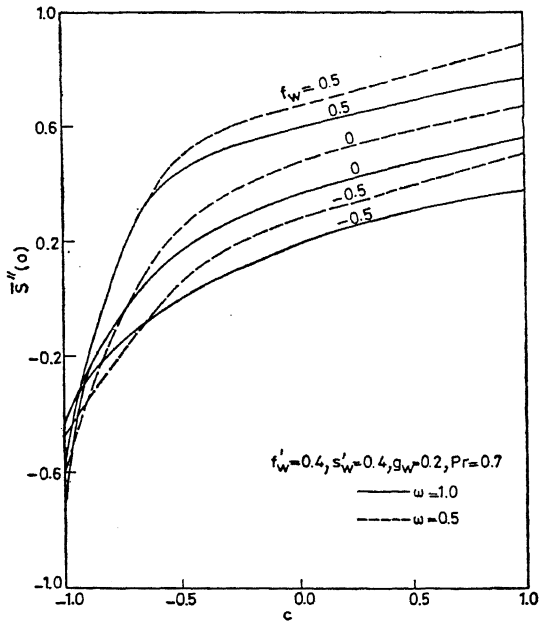

Figure 5. Effect of f_w on the variation of $\overline{S}''$ (0) with c.

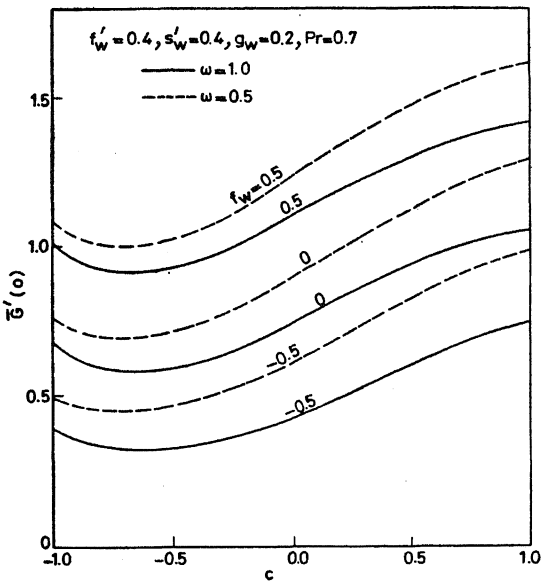

Figure 6. Effect of f_w on the variation of $\bar{G}'$ (0) with c.

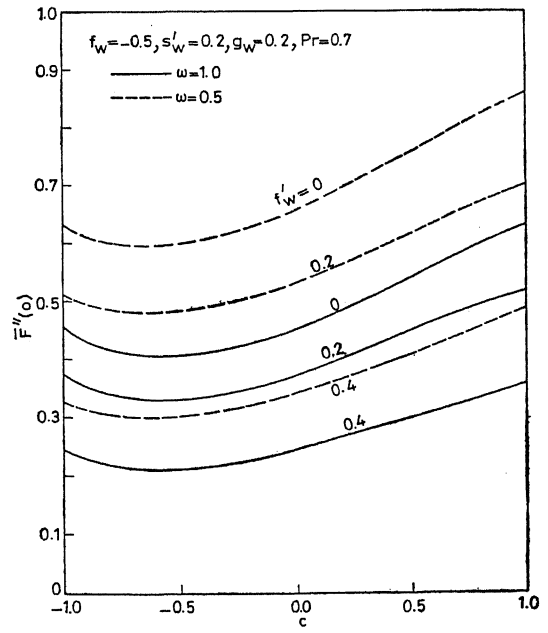

Figure 7. Effect of f'_w on the variation of $\bar{F}''$ (0) with c.

$\bar{F}''(0)$. As observed before, $\bar{F}''(0)$ and $\bar{G}'(0)$ decrease as c decreases till at some critical negative value of c, they start increasing. The physical reason for this behaviour is given by Wortman *et al* [16] hence it is not repeated here. Finally, figures 10–12 show the effect of s'_w, the surface mass transfer velocity in the transverse direction

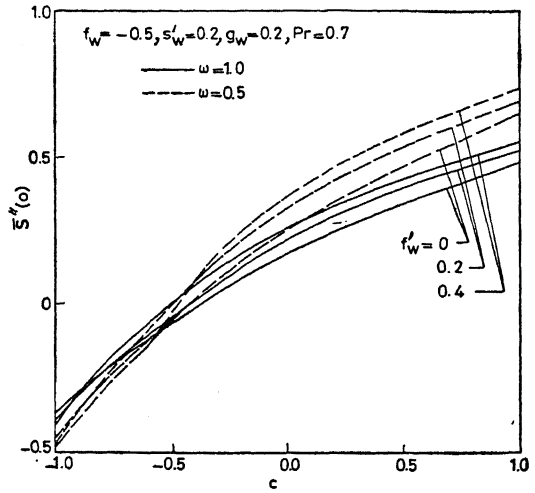

Figure 8. Effect of f'_w on the variation of $\bar{S}''(0)$ with c.

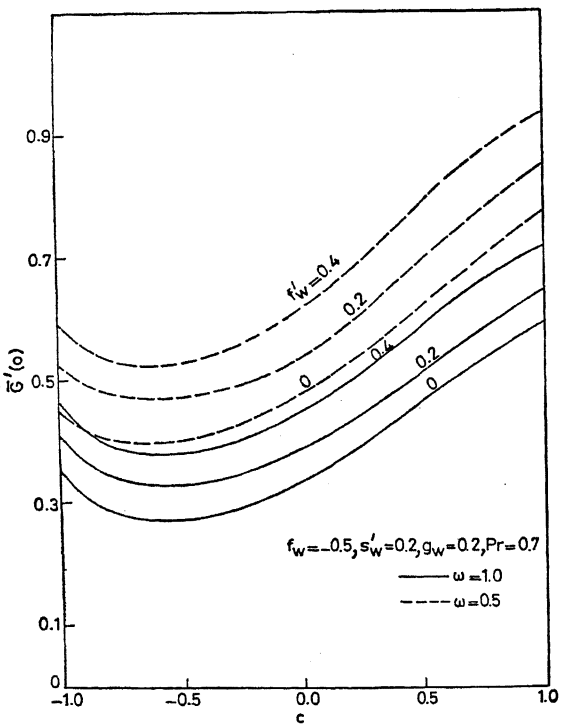

Figure 9. Effect of f'_w on the variation of $\bar{G}'(0)$ with c.

on the flow parameters. An increase in s'_w results in a decrease in $\bar{S}''(0)$ but $\bar{F}''(0)$ and $\bar{G}'(0)$ increase or decrease depending on the values of c. As expected, we observe that s'_w does not affect $\bar{F}''(0)$ and $\bar{G}'(0)$ at $c = 0$. Further, we find that for nodal points of attachment ($c > 0$) the effect of increasing s'_w results in an increase

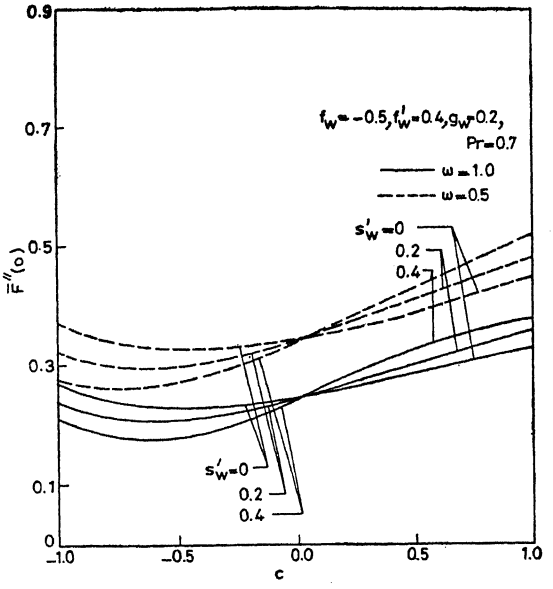

Figure 10. Effect of s'_w on the variation of $\bar{F}''$ (0) with c.

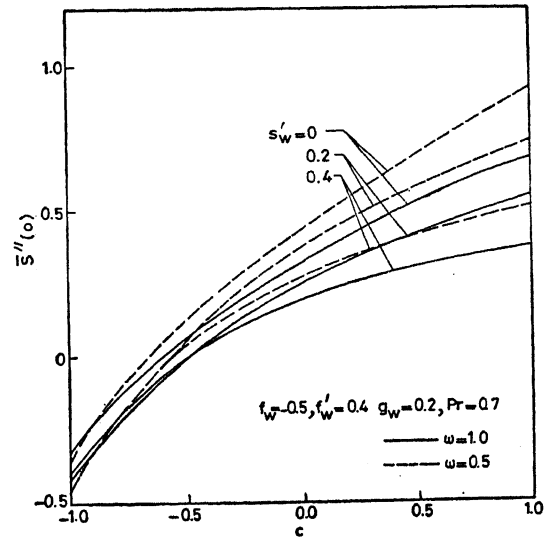

Figure 11. Effect of s'_w on the variation of $\bar{S}''$ (0) with c.

in $\bar{F}''(0)$ and $\bar{G}'(0)$, but at saddle points of attachment ($c < 0$) the reverse takes place. This is true, whatever may be the value of f_w, ω or f'_w.

On observing figures 4–12, we find that when $\omega = 0{\cdot}5$, we obtain higher values for $\bar{F}''(0)$, $\bar{S}''(0)$ and $\bar{G}'(0)$. In some cases (figures 4–9) the difference between $\omega = 1$ and $\omega = 0{\cdot}5$ profiles is quite significant. This further confirms the statement that the linear viscosity-temperature relation ($\omega = 1$) is not a good relation for obtaining an accurate estimation of the skin friction and heat transfer. We have compared

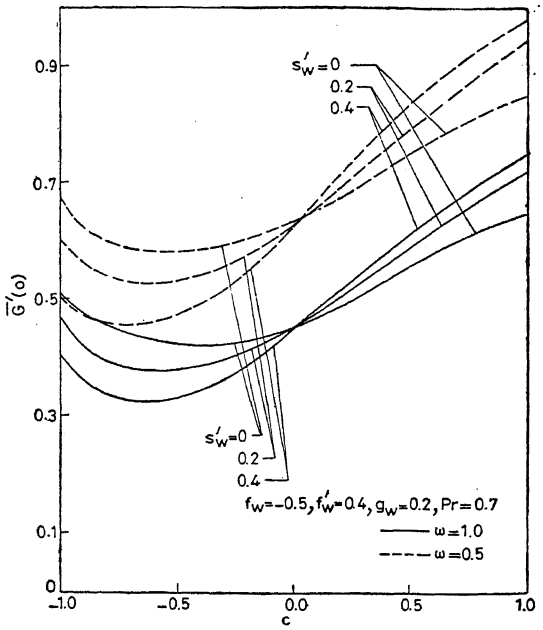

Figure 12. Effect of s'_w on the variation of $\overline{G}'$ (0) with c.

our results for $f'_w=s'_w=0$ with those of Libby [5], Wortman *et al* [16] and Vimala and Nath [14] and found them in good agreement.

4. Concluding remarks

The effect of vectored suction or injection on the skin friction and heat transfer parameters is significant and they are increased due to vectored suction while vectored injection does the reverse. The skin-friction and heat-transfer parameters are strongly dependent on the nature of the stagnation point. The effect of the variation of the density-viscosity product across the boundary layer is to increase skin friction and heat transfer and this variation gives rise to a point of inflexion in velocity and enthalpy profiles.

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Wave propagation in piezoelectric medium of hexagonal symmetry

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Abstract. The wave propagation at large distances from a source of disturbance (isolated harmonic electric charge or body force of fixed frequency) in an infinite piezoelectric medium belonging to classes (6), (6 m m) or ceramic (α m) and (6 2 2) is discussed by means of asymptotic evaluation (at large distances) of Fourier integrals. Numerical results are given for the (6 m m) crystal class using the constants of cadmium selenide crystal.

Keywords. Wave surface; wave propagation; Fourier integrals; piezoelectric medium.

1. Introduction

The geometry of wave propagation in elastic medium has been investigated by Musgrave [4], Synge [6] and others. The asymptotic behaviour (at large distances) of solutions of linear partial differential equations with constant coefficients has been demonstrated by Lighthill [3]. He has also studied the geometry of wave propagation using the asymptotic evaluation of Fourier integrals in many dimensions for large arguments. Buchwald [2] has discussed the wave propagation at large distances in elastic anisotropic medium due to radiation from a source of body force. The ' Forcing Function ' which is necessary to represent the radiation from a localized disturbance could be a body force or any other source which vanishes outside a finite region. Paldas [5] has studied the propagation of waves through piezoelectric α-quartz crystals when the source of disturbance is an isolated harmonic body force. He has studied wave surfaces due to propagation of longitudinal waves and obtained slowness surface following the analysis of Buchwald [2]. In the present paper the propagation of waves at large distances from a source of disturbance in an infinite piezoelectric medium of hexagonal symmetry belonging to classes (6), (6 m m) or ceramic (∞ m) and (6 2 2) is investigated. Two cases are analysed in which (a) the source of disturbance is an isolated harmonic electric charge of fixed frequency and (b) the source of disturbance is an isolated harmonic body force of fixed frequency. We have followed the analysis of Lighthill [3] to obtain the wave surface from the slowness surface and studied the wave amplitudes as well. Numerical results are presented for cadmium selenide crystal which belongs to (6 m m) class. It is interesting to note that the source of disturbance due to electric charge generates longitudinal waves in (6 m m) class and torsional waves in (6 2 2) class whereas the source of disturbance due to body force generates flexural waves in both (6 m m) class and (6 2 2) class.

2. Source of electric charge

We choose a rectangular coordinate system with the axis of symmetry of (6) crystal class as z-axis. The stresses T_{ij} and electric displacements D_i for crystal class (6) can be expressed in terms of displaceemnt components (u, v, w) and electric potential ϕ as

$$\begin{aligned}
T_{xx} &= c_{11}\,u_{,x} + c_{12}v_{,y} + c_{13}w_{,z} + e_{31}\phi_{,z}\\
T_{yy} &= c_{12}u_{,x} + c_{11}v_{,y} + c_{13}w_{,z} + e_{31}\phi_{,z}\\
T_{zz} &= c_{13}\,(u_{,x} + v_{,y}) + c_{33}w_{,z} + e_{33}\phi_{,z}\\
T_{yz} &= c_{44}\,(v_{,z} + w_{,y}) + e_{14}\phi_{,x} + e_{15}\phi_{,y}\\
T_{xz} &= c_{44}\,(u_{,z} + w_{,x}) + e_{15}\phi_{,x} - e_{14}\phi_{,y}\\
T_{xy} &= c_{66}\,(u_{,y} + v_{,x})\\
D_x &= e_{14}\,(w_{,y} + v_{,z}) + e_{15}\,(w_{,x} + u_{,z}) - \epsilon_{11}\phi_{,x}\\
D_y &= e_{15}\,(w_{,y} + v_{,z}) - e_{14}\,(w_{,x} + u_{,z}) - \epsilon_{11}\phi_{,y}\\
D_z &= e_{31}\,(u_{,x} + v_{,y}) + e_{33}\,w_{,z} - \epsilon_{33}\phi_{,z}
\end{aligned} \tag{1}$$

where c_{ij}, e_{ij} and ϵ_{ij} are elastic, piezoelectric and dielectric constants respectively. Comma followed by subscript denotes partial differentiation with respect to the subscript variable. Also $c_{66} = (c_{11} - c_{12})/2$.

The equations of motion (in the absence of a body force) are

$$\begin{aligned}
&c_{11}\,u_{,xx} + c_{66}\,u_{,yy} + c_{44}\,u_{,zz} + (c_{12} + c_{66})\,v_{,xy}\\
&\quad + (c_{13} + c_{44})\,w_{,xz} + (e_{15} + e_{31})\,\phi_{,xz} - e_{14}\phi_{,yz} = su_{,tt}
\end{aligned} \tag{2}$$

$$\begin{aligned}
&(c_{12} + c_{66})\,u_{,xy} + c_{66}\,v_{,xx} + c_{11}\,v_{,yy} + c_{44}\,v_{,zz}\\
&\quad + (c_{13} + c_{44})\,w_{,yz} + (e_{15} + e_{31})\,\phi_{,yz} + e_{14}\phi_{,xz} = sv_{,tt}
\end{aligned} \tag{3}$$

$$\begin{aligned}
&(c_{13} + c_{44})\,(u_{,xz} + v_{,yz}) + c_{44}\,(w_{,xx} + w_{,yy}) + c_{33}w_{,zz}\\
&\quad + e_{15}\,(\phi_{,xx} + \phi_{,yy}) + e_{33}\phi_{,zz} = sw_{,tt}
\end{aligned} \tag{4}$$

where s is the density of the medium.

When electric charge is present, Gauss's equation

Div $\mathbf{D} = 4\pi q$ gives

$$\begin{aligned}
&(e_{15} + e_{31})\,u_{,xz} - e_{14}\,u_{,yz} + e_{14}\,v_{,xz} + (e_{15} + e_{31})\,v_{,yz} + e_{15}\,(w_{,xx} + w_{,yy})\\
&\quad + e_{33}\,w_{,zz} - \epsilon_{11}\,(\phi_{,xx} + \phi_{,yy}) - \epsilon_{33}\,\phi_{,zz} = 4\pi q
\end{aligned} \tag{5}$$

From (2) and (3) we get

$$a_4 \nabla_1^2 \Lambda + a_5 \Lambda_{,zz} + a_9 \nabla_1^2 \Sigma = \Lambda_{,tt} \tag{6}$$

$$a_1 \nabla_1^2 \triangle + a_5 \triangle_{,zz} + a_3 \nabla_1^2 \Gamma + a_8 \nabla_1^2 \Sigma = \triangle_{,tt}. \tag{7}$$

From (4) and (5) we obtain

$$a_3 \triangle_{,zz} + a_5 \nabla_1^2 \Gamma + a_2 \Gamma_{,zz} + a_6 \nabla_1^2 \Sigma + a_7 \Sigma_{,zz} = \Gamma_{,tt} . \tag{8}$$

$$a_9 \Lambda_{,zz} + a_8 \triangle_{,zz} + a_6 \nabla_1^2 \Gamma + a_7 \Gamma_{,zz} - a_{10} \nabla_1^2 \Sigma - a_{11} \Sigma_{,zz} = 4\pi q_{,z} . \tag{9}$$

In eqs (6) to (9) we have taken

$$\Lambda = (v_{,x} - u_{,y}); \quad \triangle = (u_{,x} + v_{,y});$$

$$\Gamma = w_{,z} \quad ; \quad \Sigma = \phi_{,z} \quad \text{and} \tag{10}$$

$$s\,a_1 = c_{11}; \quad s\,a_2 = c_{33} \quad ; \quad s\,a_3 = c_{44} + c_{13};$$

$$s\,a_4 = c_{66}; \quad s\,a_5 = c_{44} \quad ; \quad s\,a_6 = e_{15} \quad ;$$

$$s\,a_7 = e_{33}; \quad s\,a_8 = (e_{15} + e_{31}); \quad s\,a_9 = e_{14} \quad ;$$

$$s\,a_{10} = \epsilon_{11}; \quad s\,a_{11} = \epsilon_{33}$$

The displacements u, v, w and potential ϕ are completely determined by the variables Λ, $\triangle$, Γ and Σ.

We first consider the case (case 1) when the point source is an isolated harmonic electric charge of fixed frequency ω acting at the origin; that is,

$$q = q_0 \; \delta(x) \; \delta(y) \; \delta(z) \exp(-i\,\omega\,t) \tag{11}$$

where q_0 is constant and $\delta(x)$ is Dirac's delta function.

We now express the variables Λ, $\triangle$, Γ, Σ and q in terms of Fourier integrals in the form

$$P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{P} \exp[i(\alpha x + \beta y + \gamma z - \omega t)]\, d\alpha\, d\beta\, d\gamma. \tag{12}$$

Substituting integrals of the above form for the variables in eqs (6) to (9) and considering the Fourier transform of the source term, we obtain

$$[a_4(\alpha^2+\beta^2) + a_5\gamma^2 - \omega^2]\,\overline{\Lambda} + a_9(\alpha^2+\beta^2)\,\overline{\Sigma} = 0$$

$$[a_1(\alpha^2+\beta^2) + a_5\gamma^2 - \omega^2]\,\overline{\triangle} + a_3(\alpha^2+\beta^2)\,\overline{\Gamma} + a_8(\alpha^2+\beta^2)\,\overline{\Sigma} = 0$$

$$a_3\,\gamma^2\,\overline{\triangle} + [a_5\,(\alpha^2+\beta^2) + a_2\,\gamma^2 - \omega^2]\,\overline{\Gamma} + [a_6\,(\alpha^2+\beta^2) + a_7\,\gamma^2]\,\overline{\Sigma} = 0$$

$$a_9\,\gamma^2\,\overline{\Lambda} + a_8\,\gamma^2\,\overline{\triangle} + [a_6\,(\alpha^2+\beta^2) + a_7\,\gamma^2]\,\overline{\Gamma} - [a_{10}\,(\alpha^2+\beta^2) + a_{11}\,\gamma^2]\,\overline{\Sigma} = -\,iq_0\,\gamma/(2\pi^2) \tag{13}$$

3. Slowness surface

Solutions of eq. (13) are

$$\overline{\Lambda} = K_1/H; \quad \overline{\triangle} = K_2/H; \quad \overline{\Gamma} = K_3/H \text{ and } \overline{\Sigma} = K_4/H \tag{14}$$

where

$$H \equiv H(\alpha, \beta, \gamma, \omega) = \text{det.}\,|M_{ij}|\ (i, j = 1 \text{ to } 4)$$

$$M_{11} = a_4\,(\alpha^2+\beta^2) + a_5\,\gamma^2 - \omega^2,\ M_{12} = M_{13} = 0$$

$$M_{14} = a_9\,(\alpha^2+\beta^2),\ M_{21} = 0$$

$$M_{22} = a_1\,(\alpha^2+\beta^2) + a_5\,\gamma^2 - \omega^2,\ M_{23} = a_3\,(\alpha^2+\beta^2);$$

$$M_{24} = a_8\,(\alpha^2+\beta^2),\ M_{31} = 0$$

$$M_{32} = a_3\,\gamma^2;\ M_{33} = a_5\,(\alpha^2+\beta^2) + a_2\,\gamma^2 - \omega^2$$

$$M_{34} = a_6\,(\alpha^2+\beta^2) + a_7\,\gamma^2;\ M_{41} = a_9\,\gamma^2;\ M_{42} = a_8\,\gamma^2$$

$$M_{43} = M_{34};\ M_{44} = -\,[a_{10}\,(\alpha^2+\beta^2) + a_{11}\,\gamma^2] \tag{15}$$

and $K_i \equiv K_i\,(\alpha, \beta, \gamma, \omega)$ is the determinant obtained from H, by replacing ith column with the column whose elements are respectively 0, 0, 0, $-i\,q_0\,\gamma/(2\pi^2)$. From (14) and (12), we find

$$\Lambda = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (K_1/H)\,\exp\left[i\,(\alpha x+\beta y+\gamma z-\omega t)\right]\,d\alpha\,d\beta\,d\gamma. \tag{16}$$

Similar integrals hold for $\triangle$, Γ and Σ, the integrand containing (K_i/H) $(i = 2, 3, 4$ respectively) instead of K_1/H. The only singularities of the integrand are the roots of $H = 0$. It is known (Buchwald [2]) that for large arguments, important contributions in evaluating integrals (16) arise only from points on the real surface $H = 0$ in α, β, γ space (α-space). Further outward travelling waves from source will have large contribution to the integrals. $H = 0$ regarded as an equation of α, β, γ gives the slowness surface for Λ, $\triangle$, Γ and Σ for real values of α, β, γ. We find that it is a surface of revolution. The plane $\gamma = 0$ is a plane of symmetry and γ-axis is the

axis of revolution. This surface can be represented as follows: Consider the section of $H = 0$ with plane $\beta = 0$ given by

$$H(\alpha, 0, \gamma) = 0. \tag{17}$$

Since H is homogeneous in α, β, γ and ω we can put $\gamma = k\alpha$ where k is a constant. The resulting quartic equation in α^2 can be solved. One of the four roots is found to be zero and the corresponding sheet of the surface collapses to a point viz. the origin. This corresponds to a wave with infinite velocity generated at the origin. Corresponding to each real positive root of the quartic, one real sheet of the surface is obtained. By giving arbitrary values of k at suitable intervals, we obtain a set of values $(\alpha/\omega, \gamma/\omega)$ corresponding to each non-zero positive root, which when plotted on the plane $\beta = 0$ gives a curve. Rotating this curve about γ-axis, a sheet of the slowness surface is obtained. Hence we find that in a piezoelectric medium of class (6), the slowness surface has in general three sheets.

In the case of piezoelectric medium of class (6 m m) only $a_9 = 0$. Corresponding to eqs (14) we have

$$\begin{aligned}
\bar{\Lambda} &= 0 \\
\bar{\Delta} &= [i\, q_0/(2\pi^2)]\, \gamma\, (\alpha^2 + \beta^2)\, \{(a_5 a_8 - a_3 a_6)\, (\alpha^2 + \beta^2) \\
&\quad + (a_2 a_8 - a_3 a_7)\, \gamma^2 - a_8\, \omega^2\}/H_1 \\
\bar{\Gamma} &= [i\, q_0/(2\pi^2)]\, \gamma\, \{[a_1 (\alpha^2 + \beta^2) + a_5\, \gamma^2 - \omega^2] \\
&\quad \times [a_6 (\alpha^2 + \beta^2) + a_7\, \gamma^2] - a_3 a_8 (\alpha^2 + \beta^2)\, \gamma^2\}/H_1 \\
\bar{\Sigma} &= [i\, q_0/(2\pi^2)]\, \gamma\, H_2/H_1
\end{aligned} \tag{18}$$

where H_1 is det $|\, M_{ij}\, |$, $(i, j = 2, 3, 4)$

and
$$\begin{aligned}
H_2 &= a_3^2 (\alpha^2 + \beta^2)\, \gamma^2 - [a_1 (\alpha^2 + \beta^2) + a_5\, \gamma^2 - \omega^2] \\
&\quad \times [a_5 (\alpha^2 + \beta^2) + a_2\, \gamma^2 - \omega^2].
\end{aligned} \tag{19}$$

We conclude that rotational wave Λ does not propagate in the medium and the waves of Δ, Γ, and Σ are coupled, the slowness surface being $H_1 = 0$. We find that this surface has two real sheets corresponding to the two real positive roots of the quadratic equation in α^2 given by $H_1(\alpha, 0, \gamma) = 0$.

If we consider piezoelectric medium of class (6 2 2) we have $a_6 = a_7 = a_8 = 0$ and $a_9 \neq 0$. In this case, eq. (18) becomes

$$\begin{aligned}
\bar{\Lambda} &= [i\, q_0/(2\pi^2)]\, a_9\, \gamma\, (\alpha^2 + \beta^2)/H_3 \\
\bar{\Delta} &= \bar{\Gamma} = 0 \text{ and} \\
\bar{\Sigma} &= [i\, q_0/(2\pi^2)]\, \gamma\, H_4/H_3
\end{aligned} \tag{20}$$

where

$$H_3 = [a_4(\alpha^2+\beta^2) + a_5\gamma^2 - \omega^2][a_{10}(\alpha^2+\beta^2) + a_{11}\gamma^2] + a_9{}^2(\alpha^2+\beta^2)\gamma^2. \tag{21}$$

and $$H_4 = a_4(\alpha^2+\beta^2) + a_5\gamma^2 - \omega^2$$

We find that waves of dilation $\triangle$ and Γ do not propagate. Also Λ and Σ have the slowness surface $H_3 = 0$ which has only one sheet.

4. Wave surface

The wave surface is the envelope (at time $t = 1$) of plane waves originating from a source at the origin at time $t = 0$. A point (α, β, γ) of α-space corresponds to a plane wave passing through a point (x, y, z) of x-space provided the vector (x, y, z) is parallel to the normal at (α, β, γ) to the slowness surface and the parametric equation of the wave surface is

$$x = -H_{,\alpha}/H_{,\omega};\ y = -H_{,\beta}/H_{,\omega};\ z = -H_{,\gamma}/H_{,\omega}. \tag{22}$$

We obtain the wave surface corresponding to the slowness surface $H_1 = 0$. This is the surface of revolution obtained by rotating about the z-axis in plane $y = 0$, the curve

$$Rx = \alpha\omega[b_1\alpha^4 - b_2\alpha^2\gamma^2 - b_3\gamma^4 - b_4\alpha^2\omega^2 - b_5\gamma^2\omega^2 + a_{10}\omega^4] \tag{23}$$

$$Rz = \gamma\omega[\tfrac{1}{2}b_2\alpha^4 - 2b_3\alpha^2\gamma^2 + b_6\gamma^4 - b_5\alpha^2\omega^2 - b_7\gamma^2\omega^2 + a_{11}\omega^4]$$

$$R = \omega^2\{a_8{}^2\alpha^2\gamma^2 + (a_6\alpha^2 + a_7\gamma^2)^2 + (a_{10}\alpha^2 + a_{11}\gamma^2)[(a_1+a_5)\alpha^2 + (a_2+a_5)\gamma^2 - 2\omega^2]\}$$

where

$$b_1 = 3a_1(a_5a_{10} + a_6{}^2)$$

$$b_2 = 2[2a_6(a_3a_8 - a_7a_1) - a_5a_6{}^2 - a_5(a_5a_{10} + a_1a_{11} + a_8{}^2) - a_{10}(a_1a_2 - a_3{}^2)]$$

$$b_3 = a_7[2(a_3a_8 - a_5a_6) - a_1a_7] - a_2(a_5a_{10} + a_1a_{11} + a_8{}^2) - a_{11}(a_5{}^2 - a_3{}^2) \tag{24}$$

$$b_4 = 2[a_{10}(a_1+a_5) + a_6{}^2]$$

$$b_5 = 2a_6a_7 + a_5a_{11} + a_2a_{10} + a_5a_{10} + a_1a_{11} + a_8{}^2$$

$$b_6 = 3a_5(a_2a_{11} + a_7{}^2)$$

and $$b_7 = 2[a_{11}(a_2+a_5) + a_7{}^2]$$

The method of Lighthill [3] gives us the asymptotic contributions (in 6 m m) medium for $\triangle$, Γ and Σ for large r as

$$\triangle \sim (2/r) \sum_{n=1}^{N} A_n q_0 i \lambda_n \gamma_n(\alpha_n^2 + \beta_n^2) \times$$

$$[(a_8a_5 - a_3a_6)(\alpha_n^2 + \beta_n^2) + (a_2a_8 - a_3a_7)\gamma_n^2 - a_8\omega^2]$$

$$\Gamma \sim (2/r) \sum_{n=1}^{N} A_n q_0 i\lambda_n \gamma_n \{ [a_1(\alpha_n^2 + \beta_n^2) + a_5 \gamma_n^2 - \omega^2]$$

$$\times [a_6(\alpha_n^2 + \beta_n^2) + a_7\gamma_n^2) - a_3a_8(\alpha_n^2 + \beta_n^2)\gamma_n^2\} \tag{25}$$

$$\Sigma \sim (2/r) \sum_{n=1}^{N} A_n q_0 i \lambda_n\gamma_n \{ a_3^2(\alpha_n^2 + \beta_n^2)\gamma_n^2 -$$

$$- [a_1(\alpha_n^2 + \beta_n^2) + a_5 \gamma_n^2 - \omega^2] [a_6(\alpha_n^2 + \beta_n^2) + a_7 \gamma_n^2] \}$$

where summation is taken for points $(\alpha_n, \beta_n, \gamma_n)$ on the slowness surface at which the normal is parallel to OP, the point $P(x, y, z)$ lying on the wave surface. A_n is phase constant. For an axisymmetric case, it is enough if we evaluate amplitude coefficient λ_n corresponding to $\beta=0$. Then λ_n^2 is obtained as

$$\lambda_n^2 = (H_{,\alpha}^2 + H_{,\gamma}^2)/[H_{,\beta\beta}(H_{,\alpha}^2 H_{,\gamma\gamma} - 2H_{,\alpha} H_{,\gamma} H_{,\alpha\gamma} + H_{,\gamma}^2 H_{,\alpha\alpha})] \tag{26}$$

For (6 2 2) medium the asymptotic contributions for Λ, Σ are

$$\Lambda \sim (2/r) \sum_{n=1}^{N} A_n q_0 i \lambda_n a_9 \gamma_n (\alpha_n^2+\beta_n^2)$$

and $$\sum \sim (2/r) \sum_{n=1}^{N} A_n q_0 i \lambda_n \gamma_n [a_4(\alpha_n^2+\beta_n^2)+a_5 \gamma_n^2-\omega^2].$$

5. Source of body force

The source of disturbance could be a body force instead of an electric charge. We now consider (case 2) an isolated harmonic body force of fixed frequency acting at the origin along the x-axis, the electric charge q being absent. The body force is given by

$$X = X_0 \delta(x) \delta(y) \delta(z) \exp(-i\omega t) \tag{27}$$

$$Y = Z = 0.$$

Gauss's equation in this case is Div $\mathbf{D}=0$ and eqs (13) are modified as

$$[a_4(\alpha^2+\beta^2)+a_5\gamma^2-\omega^2]\,\overline{\Lambda}+a_9(\alpha^2+\beta^2)\overline{\Sigma}=-i\beta X_0/(8\pi^3)$$

$$[a_1(\alpha^2+\beta^2) + a_5\gamma^2-\omega^2]\,\overline{\triangle} + a_3(\alpha^2 + \beta^2)\,\overline{\Gamma}$$

$$+ a_8(\alpha^2 + \beta^2)\,\overline{\Sigma} = i\alpha X_0/(8\pi^3) \tag{13^2}$$

$$a_3\gamma^2\,\overline{\triangle} + [a_5(\alpha^2+\beta^2) + a_2\gamma^2 - \omega^2]\,\overline{\Gamma} + [a_6(\alpha^2+\beta^2) + a_7\gamma^2)\,\overline{\Sigma} = 0$$

$$a_9\gamma^2\,\overline{\Lambda} + a_8\gamma^2\,\overline{\triangle} + [a_6(\alpha^2+\beta^2) + a_7\gamma^2)\,\overline{\Gamma} - [a_{10}(\alpha^2+\beta^2) + a_{11}\gamma^2)\,\overline{\Sigma} = 0.$$

Solving the above equations we obtain

$$\overline{\Lambda} = L_1/H;\ \overline{\triangle} = L_2/H;\ \overline{\Gamma} = L_3/H \text{ and } \overline{\Sigma} = L_4/H \tag{28}$$

where $L_i \equiv L_i(\alpha, \beta, \gamma, \omega)$ is the determinant obtained from H by replacing ith column with the column whose elements are respectively $-i\beta X_0/(8\pi^3)$, $i\alpha X_0/(8\pi^3)$, 0, 0.

If we consider the medium of class (6), the slowness surface is the same as in case 1.

In (6 m m) piezoelectric medium we obtain

$$\overline{\Lambda} = -i\beta X_0(8\pi^3 H_4)$$

$$\overline{\triangle} = -i\alpha X_0\{[a_5(\alpha^2+\beta^2) + a_2\gamma^2 - \omega^2][a_{10}(\alpha^2+\beta^2) + a_{11}\gamma^2)] + [a_6(\alpha^2+\beta^2) + a_7\gamma^2]^2\}/(8\pi^3 H_1) \tag{29}$$

$$\overline{\Gamma} = i\alpha X_0\gamma^2[(a_3a_{10} + a_6a_8)(\alpha^2+\beta^2) + (a_3a_{11} + a_7a_8)\gamma^2]/(8\pi^3 H_1)$$

and

$$\overline{\Sigma} = i\alpha X_0\gamma^2[(a_3a_6 - a_5a_8)(\alpha^2+\beta^2) + (a_3a_7 - a_2a_8)\gamma^2 + a_8\omega^2]/(8\pi^3 H_1).$$

The slowness surface for Λ exists and is given by $H_4=0$. In this case the slowness surface is identical with that of transversely isotropic elastic medium. Here the slowness surface is a spheroid and the corresponding wave surface is also a spheroid. The slowness surface for $\triangle$, Γ and Σ is the same as in case 1.

The asymptotic contributions for large r can be obtained as in case 1.

Considering medium of class (6 2 2), we obtain

$$\overline{\Lambda} = i\beta X_0[a_{10}(\alpha^2+\beta^2) + a_{11}\gamma^2]/(8\pi^3 H_3)$$

$$\overline{\triangle} = i\alpha X_0[a_5(\alpha^2+\beta^2) + a_2\gamma^2 - \omega^2]/(8\pi^3 H_2)$$

$$\overline{\Gamma} = i\alpha X_0 a_3\gamma^2/(8\pi^3 H_2), \text{ and} \tag{30}$$

$$\overline{\Sigma} = i\beta X_0 a_9\gamma^2/(8\pi^3 H_3)$$

Λ and Σ have the same slowness surface as in case 1. The surface for $\triangle$ and Γ exists and is given by $H_2 = 0$. The asymptotic contributions for large r can be obtained as in case 1.

6. Numerical results

Since all the constants for a class (6) crystal are not available we shall consider (6 m m) piezoelectric medium in this section. We choose the surface $H_1 = 0$ for obtaining numerical results and use the constants of cadmium selenide crystal (Berlincourt *et al* [1])

$$\begin{aligned} c_{11} &= 7{\cdot}41 \times 10^{10}\ \mathrm{N/m^2} & e_{15} &= -0{\cdot}138\ \mathrm{coul/m^2} \\ c_{12} &= 4{\cdot}51 & e_{31} &= -0{\cdot}159 \\ c_{13} &= 3{\cdot}93 & e_{33} &= -0{\cdot}347 \\ c_{33} &= 8{\cdot}36 & s &= 5684\ \mathrm{kg/m^3} \\ c_{44} &= 1{\cdot}32 & \epsilon_{11} &= 84{\cdot}4 \times 10^{-12}\ \mathrm{F/m} \\ & & \epsilon_{33} &= 90{\cdot}3 \end{aligned} \tag{31}$$

We consider the section of the surface $H_1(\alpha, \beta, \gamma)=0$ with the plane $\beta=0$. In $H_1(\alpha, 0, \gamma)=0$ we put $\gamma=k\alpha$ and evaluate the non-zero roots of the resulting quadratic equation in α^2 for different values of k^2. The corresponding values of γ^2 are obtained.

Table 1. Piezoelectric case

Coordinates of slowness surface with $\beta=0$ $\times 10^{-4}$ sec/met		Coordinates of wave surface with $y=0$ $\times 10^3$ met/sec		Amplitude coefficient $\times 10^{13}$ kg²/(Sec Coul)²
α/ω	γ/ω	x	z	λ
		First sheet		
0·00	2·59	0·00	3·86	6·6
1·09	2·44	1·04	3·63	5·2
1·23	2·40	1·21	3·54	4·9
1·36	2·35	1·38	3·45	4·7
1·46	2·31	1·52	3·37	4·5
1·67	2·20	1·83	3·16	4·2
1·98	1·98	2·34	2·70	3·9
2·04	1·93	2·44	2·61	3·8
2·10	1·87	2·54	2·50	3·8
2·16	1·81	2·65	2·37	3·8
2·66	0·84	3·46	0·93	4·3
2·77	0·00	3·61	0·00	4·6
		Second sheet		
0·00	6·56	0·00	1·52	0·2
2·37	5·30	1·34	1·29	0·8
2·60	5·06	1·40	1·26	1·1
2·79	4·84	1·42	1·25	1·8
2·95	4·66	1·43	1·24	∞
3·77	3·77	1·32	1·34	1·7
3·87	3·67	1·30	1·36	1·8
3·98	3·56	1·28	1·38	1·9
4·04	3·50	1·27	1·39	2·0
4·55	3·05	1·24	1·43	∞
5·80	1·83	1·38	1·09	0·8
6·51	0·00	1·54	0·00	0·5

Table 2. Elastic case

Coordinates of slowness surface with $\beta=0$ all $\times 10^{-4}$ sec/met		Coordinates of wave surface with $y=0$ all $\times 10^{3}$ met/sec		Amplitude coefficient all $\times 10^{-8}$ sec^2/met^2
α/ω	γ/ω	x	z	λ
		First sheet		
0·00	2·61	0·00	3·85	6·4
1·21	2·42	1·23	3·53	5·3
1·40	2·34	1·47	3·39	5·0
1·46	2·31	1·55	3·34	4·9
1·59	2·25	1·73	3·22	4·7
1·76	2·15	1·98	3·03	4·5
1·98	1·98	2·34	2·70	4·2
2·16	1·81	2·64	2·38	4·0
2·23	1·73	2·76	2·23	4·0
2·44	1·40	3·12	1·69	4·0
2·66	0·84	3·46	0·93	4·2
2·77	0·00	3·61	0·00	4·4
		Second sheet		
0·00	6·56	0·00	1·52	1·5
2·59	5·17	1·34	1·26	5·0
2·89	4·84	1·40	1·23	8·7
2·99	4·73	1·41	1·22	11·5
3·18	4·50	1·42	1·22	∞
3·43	4·21	1·41	1·23	15·2
3·80	3·80	1·35	1·28	26·9
4·01	3·59	1·32	1·31	24·1
4·28	3·32	1·28	1·36	15·4
4·80	2·84	1·26	1·39	∞
5·81	1·84	1·37	1·12	4·0
6·56	0·00	1·52	0·00	2·3

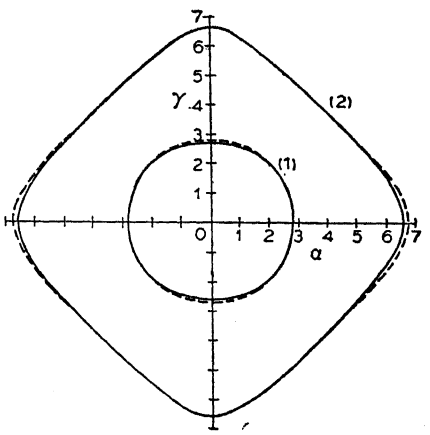

Figure 1. Slowness surfaces obtained by rotating the above curves about the γ-axis. (1) First sheet; (2) Second sheet.

x and z are found with the help of eqs (23). The amplitude coefficient λ_n is obtained from eq. (26). The results for the two sheets in the case of piezoelectric medium are given in table 1. The corresponding results, treating the medium as elastic, are given in table 2. The numerical results are also represented graphically in figures 1 and 2.

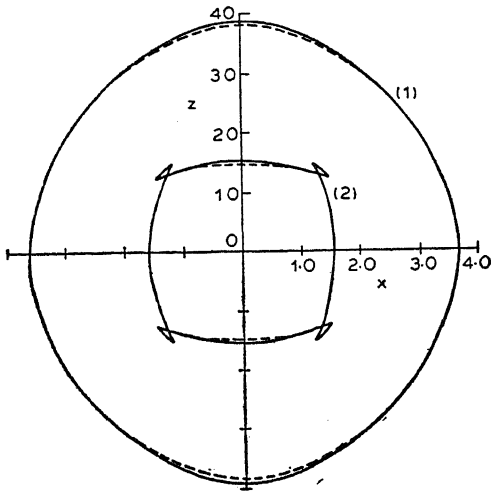

Figure 2. Wave surfaces obtained by rotating the above curves about the z-axis. (1) First sheet; (2) Second sheet.

The portions of curves where results for elastic medium are different from piezoelectric medium are shown by dotted line.

7. Conclusions

Case 1: Source of electric charge: Rotational wave Λ is not propagated in (6 m m) piezoelectric medium whereas $\triangle$, Γ and Σ waves possess a slowness surface of two sheets. The waves $\triangle$ and Γ are not propagated in (6 2 2) piezoelectric medium whereas Λ and Σ waves have a slowness surface of only one sheet. We put $\gamma = 0$ in (18) and (20) and find that $\overline{\Lambda}$, $\overline{\triangle}$, $\overline{\Gamma}$ and $\overline{\Sigma}$ are zero in both media. Hence there is no wave propagation in the plane $z = 0$.

Case 2: Source of body force: Λ wave has a slowness surface in (6 m m) piezoelectric medium and $\triangle$, Γ and Σ waves have the same slowness surface as in case 1. The waves $\triangle$ and Γ have a slowness surface in (6 2 2) piezoelectric medium. We note that in both the cases where the slowness surface does not exist for the source of electric charge, the surface obtained for the source of body force is the same as in elastic (6 m m) and (6 2 2) media. Λ and Σ in (6 2 2) medium have the same slowness surface as in case 1. We put $\alpha=\beta=0$ in equations (29) and (31) and find that $\overline{\Lambda}$, $\overline{\triangle}$, $\overline{\Gamma}$ and $\overline{\Sigma}$ are all zero. Therefore there is no wave propagation along the z-axis for both the media. The numerical results obtained for cadmium selenide (6 m m) medium enable us to observe the following. The slowness and wave surfaces are very nearly the same when the medium is considered piezoelectric or purely elastic. However, the amplitude coefficients vary considerably in the two media. Hence the electromechanical interaction does not affect much the slowness and wave surfaces but has considerable influence on the amplitude coefficient. It may be pointed out that the change will be appreciable if the medium of piezoelectric ceramic is considered instead of that of piezoelectric crystal since the electromechanical coupling factor for ceramic (∞ m) is large compared to a crystal. It may be mentioned that the

piezoelectric constants noted in the table given in [5] are wrong. Hence numerical results given in [5] cannot be relied upon.

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Magneto-elastic torsional waves in a bar under initial stress

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Abstract. The magneto-elastic torsional wave in a bar under initial stress is studied. Most of the equations have been formulated on Biot's incremental deformation theory and the initial stress has been taken in the form of uniform tension. Two cases have been considered: first when the material of the rod is homogeneous and second when it is non-homogeneous. In both the cases frequency equations for the wave have been calculated.

Keywords. Stress components; strain components; displacement component.

1. Introduction

Kolsky [7] studied the lower mode of Pochhammer's frequency equation for torsional wave and the result was that it is non-dispersive and coincides with shear wave velocity. Davies and Owen [4] investigated the higher modes of torsional wave. Jones [6] pointed out the importance of higher modes near the wave front of a disturbance and in the velocity of the source. Clark [2] discussed the torsional wave propagation in a hollow cylindrical bar. Chakravorti [3] investigated the propagation of torsional wave in a perfectly conducting elastic cylinder and a tube under the influence of a uniform axial magnetic field. In this paper magneto-elastic torsional wave propagation in a solid rod of circular cross-section under initial stress has been discussed. Most of the equations have been formulated on Biot's incremental deformation theory. The initial stress is taken in the form of a uniform tension. Frequency equation for the phase velocity of torsional wave have been calculated. Two cases have been discussed: first when material of the rod is homogeneous and second when it is non-homogeneous. The non-homogeneity of the rod is due to the variable shear modulus μ and variable density ρ. Lastly frequency equations for the said wave have been calculated. Thus several papers come out as a particular case of our paper.

2. Basic equations

The problem being one of magnetoelasticity the basic equations are those of electromagnetism and of elasticity. The Maxwell equations governing the electro-magnetic field are:

$$\text{Curl } \mathbf{H} = 4\pi \mathbf{J} \quad , \quad \text{Curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\text{div } \mathbf{B} = 0 \quad , \qquad \mathbf{B} = \mu_e \mathbf{H} \tag{1}$$

where the displacement current is neglected and Gaussian units have been used. Also, by Ohm's law,

$$\mathbf{J} = \sigma \left\{ \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} \right\}. \tag{2}$$

In eqs (1) and (2) **H**, **B**, **E**, **J** respectively denote the magnetic intensity, magnetic induction, electric intensity and current density vectors; μ_e and σ respectively denote magnetic permeability and electrical conductivity of the body, **u** represents displacement vector in the strained state and c is the velocity of light.

In the cartesian coordinate system if a body is under initial stress $\bar{\sigma}_{33}$ along z-direction only, the equilibrium equations satisfied by incremental stresses S_{ij} and initial stresses are (Biot [1]).

$$\begin{aligned}
&\frac{\partial}{\partial x} S_{11} + \frac{\partial}{\partial y} S_{12} + \frac{\partial}{\partial z} S_{31} + \bar{\sigma}_{33} \frac{\partial \omega_y}{\partial z} + F_x = \rho \frac{\partial^2 u}{\partial t^2} \\
&\frac{\partial}{\partial x} S_{12} + \frac{\partial}{\partial y} S_{22} + \frac{\partial}{\partial z} S_{23} - \bar{\sigma}_{33} \frac{\partial \omega_x}{\partial z} + F_y = \rho \frac{\partial^2 v}{\partial t^2} \\
&\frac{\partial}{\partial x} S_{31} + \frac{\partial}{\partial y} S_{23} + \frac{\partial}{\partial z} S_{33} + \bar{\sigma}_{33} \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) + F_z = \rho \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{3}$$

where u, v, w are displacement components in incremental stage and

$$\begin{aligned}
\omega_x &= \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \\
\omega_y &= \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right).
\end{aligned} \tag{4}$$

These equilibrium equations in cylindrical polar coordinates are given by,

$$\begin{aligned}
&\frac{\partial}{\partial r} \sigma_{rr} + \frac{1}{r} \frac{\partial}{\partial \theta} \sigma_{r\theta} + \frac{\partial}{\partial z} \sigma_{rz} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \bar{\sigma}_{33} \frac{\partial \omega_\theta}{\partial z} + F_r = \rho \frac{\partial^2 u_r}{\partial t^2} \\
&\frac{\partial}{\partial r} \sigma_{r\theta} + \frac{1}{r} \frac{\partial}{\partial \theta} \sigma_{\theta\theta} + \frac{\partial}{\partial z} \sigma_{\theta z} + \frac{2\sigma_{r\theta}}{r} - \bar{\sigma}_{33} \frac{\partial \omega_r}{\partial z} + F_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2} \\
&\frac{\partial}{\partial r} \sigma_{rz} + \frac{1}{r} \frac{\partial}{\partial \theta} \sigma_{\theta z} + \frac{\partial}{\partial z} \sigma_{zz} + \frac{\sigma_{rz}}{r} + \bar{\sigma}_{33} \left(\frac{\partial \omega_\theta}{\partial r} - \frac{1}{r} \frac{\partial \omega_r}{\partial \theta} \right) + F_z = \rho \frac{\partial^2 u_z}{\partial t^2}
\end{aligned} \tag{5}$$

where σ_{ij} are incremental stress components along rotated directions of the axes and

$$\begin{aligned}
\omega_r &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \\
\omega_\theta &= \frac{1}{2} \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right)
\end{aligned} \tag{6}$$

where $\mathbf{u} = (u_r, u_\theta, u_z)$ and $\bar{\sigma}_{33}$ is the initial stress along original z-directions. Assuming that the stresses and displacements are independent of θ, the equilibrium equation takes the form,

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{\partial}{\partial z}\sigma_{rz} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\bar{\sigma}_{33}}{2}\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{\partial^2 u_z}{\partial r \partial z}\right) + F_r = \rho \frac{\partial^2 u_r}{\partial t^2}$$

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \frac{2\sigma_{r\theta}}{r} + \bar{\sigma}_{33}\frac{\partial^2 u_\theta}{\partial z^2} + F_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2} \tag{7}$$

$$\frac{\partial}{\partial r}\sigma_{rz} + \frac{\partial}{\partial z}\sigma_{zz} + \frac{\sigma_{rz}}{r} + \frac{\bar{\sigma}_{33}}{2}\left(\frac{\partial^2 u_r}{\partial r \partial z} - \frac{\partial^2 u_z}{\partial r^2}\right) + F_z = \rho \frac{\partial^2 u_z}{\partial t^2}$$

where $\mathbf{F} = [F_r, F_\theta, F_z]$ is the Lorentz force per unit volume due to the axial magnetic field and is given by,

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \tag{8}$$

The electromagnetic field equations in vacuum are,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}^* = 0 \tag{9}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{h}^* = 0 \tag{10}$$

$$\text{Curl } \mathbf{E}^* = -\frac{1}{c}\frac{\partial \mathbf{h}^*}{\partial t} \tag{11}$$

$$\text{Curl } \mathbf{h}^* = \frac{1}{c}\frac{\partial \mathbf{E}^*}{\partial t} \tag{12}$$

where $\mathbf{h}^*$ is the perturbation in the magnetic field in vacuum and $\mathbf{E}^*$ is the electric field in vacuum. Moreover, the incremental boundary forces per unit area may be written as

$$\begin{aligned}
\triangle fr &= \sigma_{rr} n_r + \sigma_{r\theta} n_\theta + \sigma_{zr} n_z + \bar{\sigma}_{33}\, \omega_\theta\, n_r \\
\triangle f\theta &= \sigma_{\theta r} n_r + \sigma_{\theta\theta} n_\theta + \sigma_{\theta z} n_z - \bar{\sigma}_{33}\, \omega_r\, n_z \\
\triangle fz &= \sigma_{zr} n_r + \sigma_{\theta z} n_\theta + \sigma_{zz} n_z + \bar{\sigma}_{33}(e_{rr} + e_{\theta\theta}) n_z \\
&\quad - \bar{\sigma}_{33}\, e_{zr}\, n_r - \bar{\sigma}_{33}\, e_{\theta z}\, n_\theta
\end{aligned} \tag{13}$$

where n_i is the cosine of the angle between normal to the deformed boundary and ith direction before incremental deformation. In case of torsional vibration, $u_r = u_z = 0$ and $u_\theta = v$.

Hence

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{zr} = 0. \tag{14}$$

Consequently, the incremental boundary forces may be written as,

$$\begin{aligned} &\triangle fr = 0 \\ &\triangle f_\theta = \sigma_{rz} + \frac{\bar{\sigma}_{33}}{2} \frac{\partial \omega_\theta}{\partial z} \\ &\triangle fz = 0 \end{aligned} \tag{15}$$

3. Consideration of the problem and method of solution

Let us consider a semi infinite solid cylindrical rod of radius under initial stress which is a tension P along the axis of the rod and suppose that the elastic properties of the rod are symmetrical about z-axis. The torsional wave is then characterized by the incremental displacement.

$$u_r = 0,\ u_z = 0 \text{ and } u_\theta = v\,(r, z) \tag{16}$$

and hence,

$$\begin{aligned} &e_{rr} = 0,\ e_{\theta\theta} = 0,\ e_{zz} = 0, \quad e_{zr} = 0 \\ &2e_{\theta z} = -\frac{\partial v}{\partial z},\ 2e_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} \\ &2\omega_r = -\frac{\partial v}{\partial z},\ 2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r} \end{aligned} \tag{17}$$

where e_{ij} are incremental strain components and ω_i are rotational components respectively. Now, let us suppose that

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h} \tag{18}$$

where $\mathbf{H}_0$ is the initial magnetic field acting parallel to z-axis and $\mathbf{h}$ is the small perturbation in the field. If the rod considered is a perfect conductor of electricity (i.e. $\sigma \to \infty$), eq. (2) gives,

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{B} = \left[-\frac{H}{c}\frac{\partial v}{\partial t},\ 0,\ 0\right] \tag{19}$$

where $H = |\mathbf{H}_0|$.

Eliminating **E** from (1) and (19) we get,

$$\mathbf{h} = \left[0,\ H\frac{\partial v}{\partial z},\ 0\right]. \tag{20}$$

From eqs (1) and (20) we have

$$\mathbf{J} \times \mathbf{B} = \mathbf{J} \times \mu_e \mathbf{H}_0 = \left[0,\ -\frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2},\ 0\right]. \tag{21}$$

Considering eqs (14), (16) and (17) we see that two equations of motion are identically satisfied and the remaining one takes the form,

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \frac{2}{r}\sigma_{r\theta} + \frac{P}{2}\frac{\partial^2 v}{\partial z^2} - \frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{22}$$

Also, the stress-strain relations are,

$$\sigma_{r\theta} = 2\mu e_{r\theta},\ \sigma_{\theta z} = \mu e_{\theta z}. \tag{23}$$

Case I—When the material of the rod is homogeneous

In this case eq. (22) with the help of (17) and (23) takes the form,

$$\mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \left(\mu + \frac{P}{2} - \frac{H^2}{4\pi}\right)\frac{\partial^2 v}{\partial z^2} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{24}$$

Considering the harmonic wave along z-axis, we take the solution of (24) as,

$$v = V(r)\exp\left[i\left(qz + pt\right)\right] \tag{25}$$

then eq. (24) reduces to,

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \left(K^2 - \frac{1}{r^2}\right)V = 0 \tag{26}$$

where

$$K^2 = \frac{\rho p^2}{\mu} - \frac{q^2\left(\mu + \frac{P}{2} - \frac{H^2}{4\pi}\right)}{\mu}. \tag{27}$$

Putting $R = Kr$ in eq. (26) we have,

$$\frac{d^2 V}{dR^2} + \frac{1}{R}\frac{dV}{dR} + \left(1 - \frac{1}{R^2}\right)V = 0. \tag{28}$$

The solution (28) is given by

$$V = A J_1 (Kr) \tag{29}$$

where A is constant and J_1 is Bessel's function of first kind and first order. Therefore the solution of (24) is

$$v = A J_1 (Kr) \exp [i (qz + pt)]. \tag{30}$$

Suppose the boundary $r = a$ of the rod which separates the rod from vacuum be free from stresses. Then the required boundary conditions are

$$\sigma_{r\theta} + T_{r\theta} = T^*_{r\theta} \text{ on } r = a$$

$$E = E^* \text{ on } r = a \tag{31}$$

where $T_{r\theta}$ and $T^*_{r\theta}$ are Maxwell stresses in the body and in vacuum respectively. We can easily verify that (Chakravorty [3])

$$T_{r\theta} = T^*_{r\theta} = 0. \tag{32}$$

Equation (31) together with (23) yields $e_{r\theta} = 0$ which with the help of (17) and (25) gives,

$$\frac{\partial}{\partial r}\left\{\frac{J_1 (Kr)}{r}\right\} = 0 \text{ at } r = a. \tag{33}$$

On simplification this gives

$$\frac{J_0 (Ka)}{J_1 (Ka)} = \frac{2}{Ka}. \tag{34}$$

Equation (34) has multiple roots; the first three being $Ka = 0$, 5·136, 8·418. If R_1 stands for any one of the roots then from eq. (27) and (34) we have

$$\frac{C_1}{\beta_1} = \left\{ \frac{R_1^2 \left(\frac{\lambda}{a}\right)^2}{4\pi^2 \left(1 + G - \frac{H^2}{4\pi\mu}\right)} + 1 \right\} \tag{35}$$

where $G = P/2\mu$, λ the wavelength $= 2\pi/q$, c_1 the phase velocity $= p/q$ and

$$\beta_1 = \left\{ \frac{\mu + \frac{P}{2} - \frac{H^2}{4\pi}}{\rho} \right\}^{1/2} \tag{36}$$

is the velocity of transverse wave under initial tension P along the direction of propagation under the influence of magnetic field of intensity **H**. Equation (35) gives the relation between phase velocity and wavelength of torsional wave for various values of R_1, G and H. The presence of G and H in eq. (35) shows the effect of initial stress and magnetic field respectively on phase velocity. If there were no magnetic field then,

$$\frac{C_2}{\beta_2} = \left\{ \frac{R_1^2 \left(\frac{\lambda}{a}\right)^2}{4\pi^2 (1 + G)} + 1 \right\}^{1/2} \tag{37}$$

where

$$\beta_2 = \left\{ \frac{\mu + \frac{P}{2}}{P} \right\}^{1/2} \tag{38}$$

which coincides with the results of Dey (5). In eqs (37) and (38) if $G = 0$ and $H = 0$, then the result coincides with the classical case of Kolsky [7].

The first root $R_1 = 0$ gives $c_1 = \beta_1$ i.e. the velocity of torsional wave coincides with shear wave velocity and propagation is non-dispersive. It is easy to verify that for other values of R_1 dispersion takes place and c_1 increases along with R_1 for fixed values of λ, a, G and H. Writing eq. (35) in the form

$$\frac{C_1}{\beta_1} = \left\{ \frac{R_1^2 \left(\frac{\lambda}{a}\right)^2}{4\pi^2} + 1 + G - \frac{H^2}{4\pi} \right\}^{1/2} \tag{39}$$

where $\beta_1 = \sqrt{(\mu/\rho)}$. This shows that c_1 increases with P. Also, from eq. (35) we see that c_1 is zero if,

$$G = -1 - \frac{R_1^2 \left(\frac{\lambda}{a}\right)^2}{4\pi^2} + \frac{H^2}{4\pi\mu}. \tag{40}$$

This indicates that torsional wave does not propagate in the medium when takes the value given by eq. (40). This happens when the material is unstable or at the point of collapse.

Case II—When the material of the rod is non-homogeneous.

We assume

$$\mu = \mu_0 r^2$$

$$\rho = \rho_0 r^2 \tag{41}$$

where μ_0, ρ_0 are constants and r is the radius vector. Thus eq. (22) with the help of (17), (23) and (41) takes the form,

$$\frac{\partial^2 v}{\partial r^2} + \frac{3}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \left\{1 + \left(\frac{P}{2\mu_0} - \frac{H^2}{4\pi\mu_0}\right)\frac{1}{r^2}\right\}\frac{\partial^2 v}{\partial z^2} = \frac{\rho_0}{\mu_0}\frac{\partial^2 v}{\partial t^2}. \tag{42}$$

Considering the harmonic wave along z-axis, we take the solution of (42) as,

$$v = V_1(r) \exp\left[i\left(q_1 z + p_1 t\right)\right] \tag{43}$$

then eq. (42) reduces to,

$$\frac{d^2 V_1}{dr^2} \frac{3}{r}\frac{dV_1}{dr} + \left(\lambda - \frac{\mu^2}{r^2}\right) V_1 = 0 \tag{44}$$

where

$$\lambda^2 = \frac{\rho_0^2 p_1^2}{\mu_0} - q_1^2$$

$$\mu^2 = \frac{P q_1^2}{2\mu_0} - \frac{H^2 q_1^2}{4\pi\mu_0} + 3.$$

Putting

$$V_1 = \frac{1}{r}\psi(r) \tag{45}$$

in (44) we obtain

$$\frac{d^2\psi}{d r^2} - \frac{1}{r}\frac{d\psi}{dr} + \left(\lambda^2 - \frac{\nu^2}{r^2}\right)\psi = 0 \tag{46}$$

where

$$\nu^2 = \mu_0^2 + 1. \tag{47}$$

The solution of eq. (46) may be given by

$$\psi(r) = A_1 J_\nu(\lambda r) + B_1 Y_\nu(\lambda r) \tag{48}$$

where A_1, B_1 are constants and J_ν, Y_ν are Bessel functions of first and second kind. Using eqs (43) and (45) we obtain

$$v = \frac{1}{r}\left\{A_1 J_\nu(\lambda r) + B_1 Y_\nu(\lambda r) \exp\left[i(q_1 z + p_1 t)\right]\right\}. \tag{49}$$

Equation (31) together with (23) yields $e_{r\theta} = 0$ which with the help of (17) and (49) gives

$$\lambda r J_{\nu-1}(\lambda r) - (\lambda\nu + 2) J_\nu(\lambda r) = 0 \text{ at } r = a. \tag{50}$$

On simplification we obtain

$$\frac{J_{\nu-1}(\lambda a)}{J_\nu(\lambda a)} = \frac{\lambda\nu+2}{\lambda a}. \tag{51}$$

Equation (51) has multiple roots. If R_2 stands for any one of the roots then from eqs (45) and (51) we have,

$$\frac{C_2}{\beta_2} = \left[\mu\left\{1 + \frac{\lambda\left(\left(\frac{R_2}{a}\right)^2 - \mu^2 - 3\right)}{4\pi^2\left(1 - G_1 - \frac{H^2a^2}{4\pi\mu}\right)}\right\}\right]^{1/2} \tag{52}$$

where $G = P/2\mu$, λ the wave length $= 2\pi/q$, $c_2 =$ the phase velocity $= p_1/q_1$, and

$$\beta_2 = \frac{a\left(1 - G - \frac{H^2a^2}{4\pi\mu}\right)^{1/2}}{\rho} \tag{53}$$

is the velocity of the transverse wave under initial tension P along the direction of propagation under the influence of magnetic field of intensity H. Equation (52) gives the relation between phase velocity and wavelength of torsional wave for various values of R_2, G_2 and H. The rest of the discussion regarding this equation are exactly similar to the previous case.

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Free and forced convection flow in a rotating channel bounded below by a permeable bed

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Abstract. The steady flow in a parallel plate channel rotating with an angular velocity Ω and bounded below by a permeable bed is analysed under the effect of buoyancy force. On the porous bed the boundary condition of Beavers and Joseph is applied and an exact solution of the governing equations is found. The solution in dimensionless form contains four parameters: The permeability parameter σ^2, the Grashof number G, the rotation parameter K^2 and a dimensionless constant α. The effects of these parameters, specially, σ^2, G and K^2, on the slip velocities and velocity distributions are studied. For large K^2, there arise thin boundary layers on the walls of the channel.

Keywords. Free convection; forced convection; permeable bed; rotating channel.

1. Introduction

A number of workers have theoretically studied flows through pipes and channels rotating with a constant angular velocity about an axis perpendicular to their length. Mention may be made of the work of Barua [1], Benton [2], Benton and Boyer [3] for a circular pipe and Vidyanidhi and Nigam [17], Nanda and Mohanty [13], Gupta [11] and Mohan [12] for a channel formed by two parallel plates.

The problems of flow over a porous bed involve two regions: the porous bed in which the flow is governed by the Darcy's law and the fluid chamber in which the flow is governed by the Navier-Stokes equations. The solutions in two flow regimes are to be matched at the interface between the fluid and porous medium. Recently, Beavers and Joseph [4] have proposed the slip velocity boundary condition at the interface as

$$\frac{dq_t}{dn} = \frac{\alpha}{\sqrt{k'}}(q_{tB} - V_t) \tag{1}$$

where t denotes the tangential component and d/dn denotes the normal derivative to the porous interface, q_{tB} is the slip-velocity and α is a dimensionless constant. The existence of this slip-velocity q_{tB} is connected with the presence of a very thin boundary layer of stream-wise moving fluid just beneath the surface of porous material.

Beavers and Joseph [4], in order to test the validity of the proposed boundary condition (1), made a theoretical and experimental study of flow in a parallel plate

channel. Owing to inadequate apparatus and instruments, the accuracy of experimental data was not sufficient to permit a conclusive evaluation of the proposed analytical model, but the existence of slip-velocity was qualitatively confirmed. Beavers *et al* [5] have undertaken to verify the slip-velocity condition (1) proposed by Beavers and Joseph [4] and explored the influence of a porous bounding wall on laminar-turbulent transition. Beavers *et al* [5] have performed the experiment in a parallel plate channel, one of whose bounding walls was porous while the other bounding wall was impermeable. They have shown that the experimental results for the laminar flow regime are in excellent accord with theoretical predictions based on a model which admits a slip-velocity at the surface of porous material.

Beavers and Joseph [4] and, later, Beavers [5] have not discussed a method of finding an analytical expression for the boundary layer thickness δ'. To derive an analytical expression for δ', we need a momentum equation similar to the boundary layer equation in the porous bed which involves both the viscous dissipation term and the Darcy's resistance term. This equation in hydrodynamics has been given by Brinkman [6] as

$$-\nabla p'+\mu\nabla^2\cdot\mathbf{q}-\frac{\mu}{k'}\mathbf{q}=0 \tag{2}$$

where $\mathbf{q}$, μ are the velocity vector and the coefficient of viscosity of the fluid respectively, k' is the permeability of the porous medium and p' is the hydrodynamic pressure in the porous medium.

An exhaustive study of free convection problems for the flow between two vertical plates has been made by Gershuni and Zhukhovitskii [7], Poots [14], Yu [18], Singer [16] and Yu and Yang [19] Rajasekhara [15] considered the problem of natural convection flow between two vertical plates one of them being bounded by a permeable bed and obtained an expression for the boundary layer thickness using (1) and (2). The effect of buoyancy forces on fully developed flow between two horizontal parallel plates was first considered by Gill and Casal [8]. The axial temperature gradient was taken constant throughout the flow system. Subsequently Gupta [10] extended it to magnetohydrodynamics.

The present paper is devoted to the study of combined free and forced convection viscous flow in a parallel plate channel bounded below by a permeable bed and rotating with an angular velocity Ω about an axis perpendicular to the length of plate. Solutions of the equations of motion and temperature for the flow through chamber and porous medium are obtained. Effects of the wall permeability and buoyancy force on the slip velocities and velocity profiles in the chamber are discussed.

2. Mathematical formulation

The hydrodynamic equations of motion, continuity and energy for an incompressible fluid, under steady state conditions, in a rotating frame are

$$\mathbf{q}\cdot\nabla\cdot\mathbf{q}+2\Omega\times\mathbf{q}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\cdot\mathbf{q}-g\{1-\beta(T-T_0)\}\hat{k} \tag{3}$$

$$\nabla \cdot \mathbf{q}=0 \tag{4}$$

$$\rho c(\mathbf{q} \cdot \nabla)T=k\nabla^2 T+\phi \tag{5}$$

and the Darcy's laws describing the flow in the permeable bed, together with the equations of continuity and energy, in a rotating frame are

$$\mathbf{V}=-\frac{k'}{\nu}\left[\frac{1}{\rho}\nabla p+2\Omega\times\mathbf{V}+g\{1-\beta(T-T_0)\}\hat{k}\right] \tag{6}$$

$$\nabla \cdot \mathbf{V}=0 \tag{7}$$

$$\rho c(\mathbf{V} \cdot \nabla)T=k_1\nabla^2 T+\frac{\mu}{k'}(\mathbf{V})^2 \tag{8}$$

and the Brinkman's eq. (2) in the same rotating frame is

$$\frac{\nu}{k'}\mathbf{q}+2\Omega\times\mathbf{q}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\cdot\mathbf{q} \tag{9}$$

where ρ, ν, c and β are, respectively, the density, kinematic coefficient of viscosity, specific heat at constant pressure and the expansivity of the fluid. $\mathbf{V}$ is the Darcy's velocity (or filtration velocity) of the fluid and ϕ is a dissipation function. k and k_1 are, respectively, the coefficients of heat conductivity of the fluid for chamber and porous medium. g is acceleration due to gravity and $p=p'-\frac{1}{2}\rho\,|\,\Omega\times\mathbf{r}\,|^2$, $\mathbf{r}$ denoting the position vector from the axis of rotation.

Consider the flow of an incompressible fluid in a horizontal rectangular channel bounded by a solid wall at the top and a porous bed at the bottom. The channel consists of other two side-walls which are solid and perpendicular to the upper and lower walls. With reference to a cartesian coordinates system, the upper wall is located at $z=L$ and the porous bed at $z=-L$ and the two side-walls (solid) are placed at $y=\pm b$. It is assumed that $b\gg L$. The whole system rotates in a counter clockwise direction about z-axis with an angular velocity Ω. It is assumed that the flow in chamber is governed by the Navier-Stokes equations and the flow in porous medium is governed by the Darcy's law and it is also assumed that the flow in chamber and in porous bed is driven by the same pressure gradient and the same buoyance force. The fluid is assumed to be Boussinesq. If we confine ourselves to the central core region far away from the vertical walls and the channel is long enough, fully developed conditions can be assumed to exist and in the steady state, the velocity and the temperature fields can be taken to depend only on z. Under these approximations, the velocity field and the temperature distribution in the chamber and in the porous bed must obey, Chamber:

$$-2v'\Omega=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u'}{\partial z^2} \tag{10}$$

$$2u'\,\Omega=\nu\frac{\partial^2 v'}{\partial z^2} \tag{11}$$

$$O=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g\{1-\beta(T-T_0)\} \tag{12}$$

$$u'\frac{\partial T}{\partial x}+v'\frac{\partial T}{\partial y}=\alpha'\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{\nu}{c}\left\{\left(\frac{\partial u'}{\partial z}\right)^2+\left(\frac{\partial v'}{\partial z}\right)^2\right\} \tag{13}$$

Porous medium:

$$U'=-\frac{k'}{\nu}\left[\frac{1}{\rho}\frac{\partial p}{\partial x}-2\Omega V'\right] \tag{14}$$

$$V'=-\frac{k'}{\nu}(2\Omega U') \tag{15}$$

$$O=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g\{1-\beta(T-T_0)\} \tag{16}$$

$$\rho c\left(U'\frac{\partial T}{\partial x}+V'\frac{\partial T}{\partial y}\right)=k_1\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{\mu}{k'}(U'^2+V'^2) \tag{17}$$

where $\mathbf{q}=(u', v', o)$, $\mathbf{V}=(U', V', O)$, $\boldsymbol{\Omega}=(O, O, \Omega)$, and α' is the thermal diffusivity.

Integrating (12) or (16) with respect to z, we get

$$\frac{1}{\rho}p=-gz+K_1x+g\beta\int(T-T_0)dz \tag{18}$$

and we describe the temperature T as

$$T=T_0+A_1x+T'(z)$$

with $T=T_0$, $A_1=0$ and $T'=0$ at $z=-L-\delta'$

where A_1 and K_1 are constants which must be related to the physics of the problem. A_1 could be determined from a consideration of heat balance on moving fluid while K_1 could be related to the pressure at the ends.

These equations are supplemented by the equation of state

$$\rho=\rho_0[1-\beta(T-T_0)] \tag{19}$$

where ρ_0 is the density at the ambient temperature T_0 and β is the expansivity of the fluid defined by

$$\beta=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p \tag{20}$$

2.1. *Boundary conditions in the chamber*

We need two boundary conditions for the velocity and the temperature. The first boundary condition on the velocity in the chamber is usual no-slip condition at the solid wall, that is,

$$u'=0, v'=0 \text{ at } z=L \tag{21}$$

and the second boundary condition on the velocity is the slip-velocity boundary condition at the permeable bed as proposed by Beavers and Joseph (4), that is,

$$\frac{dF'}{dz} = \frac{\alpha}{\sqrt{k'}}(F_B' - H') \text{ at } z = -L \tag{22}$$

where $F' = u' + iv'$, $H' = U' + iV'$, $F_B' = u_B' + iv_B'$ and u_B' and v_B' are the slip-velocities at the permeable interface ($z = -L$) in the directions of x- and y-axes respectively.

The two boundary conditions on the temperature will depend on the nature of the solid wall and the permeable interface. In the present discussion we consider a situation where the upper solid plate is maintained at a constant temperature and the permeable interface may be made up of a perfectly insulating material, a perfectly conducting material or the material to be such that there is a uniform heat flux at the interface. In what follows, we consider only a case when there is a uniform heat flux at the interface.

The condition of uniform heat flux rate at the permeable interface $z = -L$ is characterised by

$$-k\left(\frac{\partial T'}{\partial z}\right)_{z=-L} = q' \tag{23}$$

where q' is the heat transfer across the bed and is given by

$$q' = -h_T \cdot T_B' \tag{24}$$

h_T being the heat transfer coefficient and $T' = T_B'$ at $z = -L$. Therefore,

$$\frac{\partial T'}{\partial z} = \frac{h_T \cdot T_B'}{k} \text{ at } z = -L. \tag{25}$$

This boundary condition is used on the fact that the flow, which is caused by temperature differences in the fluid, builds a thin boundary layer just adjacent to the permeable interface in the porous material. Within this thin layer, the temperature decreases from the value T_B on the permeable interface to the value $T = 0$ just outside the layer i.e. outside the heat layer.

Throughout the analysis, we assume that the boundary layer thickness for velocity and temperature is the same, for this will reduce the hard task of numerical computations and the results may deviate only moderately from those with unequal boundary layer thickness.

2.2. *Boundary conditions in the porous material*

The temperature distribution in the porous bed cannot be derived until the boundary conditions are specified. However, the boundary condition (22) has been derived on the basis that there will be a thin boundary layer inside the porous bed adjacent to the interface. To determine an expression for this boundary layer thickness, eqs (14) and (15) are to be modified to include the usual viscous dissipation term of the Navier-

Stokes equation in addition to the Darcy's resistance term which is the Brinkman's [6] model. In this case, the momentum equations (14) and (15) in the bed take the form

$$\nu \frac{\partial^2 F'}{\partial z^2} - \left(\frac{\nu}{k'} + 2i\Omega\right) F' = \frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{26}$$

To determine the velocity, two boundary conditions are needed and they are

$$F' = F_B' \text{ at } z = -L \tag{27}$$

$$\frac{\partial F'}{\partial z} = 0 \text{ at } z = -\delta' - L \tag{28}$$

where δ' is the boundary layer thickness from the interface to the point at which the velocity is minimum i.e. the velocity tends to a free stream velocity.

There are two boundary conditions for the temperature and they are

$$T' = T_B' \text{ at } z = -L \tag{29}$$

$$T' = 0 \quad \text{at } z = -\delta' - L \tag{30}$$

3. Solutions

By introducing the following substitutions

$$\eta = \frac{z}{L}, \qquad u = \frac{u'L}{\nu}, \qquad v = \frac{v'L}{\nu}, \qquad \theta = \frac{g\beta L^3 T'}{\nu^2},$$

$$\theta_B = \frac{g\beta L^3 T_B'}{\nu^2}, \quad U = \frac{U'L}{\nu}, \quad V = \frac{V'L}{\nu}, \qquad \sigma = \frac{L}{\sqrt{k'}}$$

eqs (10), (11), (13), (14), (15), (17), (21), (22) and (25) to (30) can be placed in the dimensionless form to give

Chamber:

$$\frac{d^2F}{d\eta^2} - 2iK^2 F = G\eta + R \tag{31}$$

$$Gu = \frac{1}{P_r}\frac{d^2\theta}{d\eta^2} - E\left[\left(\frac{du}{d\eta}\right)^2 - \left(\frac{dv}{d\eta}\right)^2\right] \tag{32}$$

with the boundary conditions

$$F = 0, \ \theta = \theta_0 \text{ (say) at } \eta = 1 \tag{33}$$

$$\frac{dF}{d\eta} = \alpha\,\sigma(F_B - H) \text{ at } \eta = -1 \tag{34}$$

$$\frac{d\theta}{d\eta} = N_u \cdot \theta_B \text{ at } \eta = -1 \tag{35}$$

where $F = u + iv$, $F_B = u_B + iv_B$ and $H = U + iV$, $G = g\beta A_1 L^4/\nu^2$ is the Grashof number, $E = (g\beta L/c)$ is the Eckert number, $R = (L^3 K_1)/\rho\nu^2$, $K^2 = \Omega L^2/\nu$ is the rotation parameter, $P_r = \mu c/k$ is the Prandtl number and $N_u = (h_T \cdot L)/k$ is the Nusselt number.

Porous medium:

$$H = -\frac{(G\eta + R)}{\sigma^2 + 2iK^2} \tag{36}$$

$$GU = \frac{1}{P_r'}\frac{d^2\theta}{d\eta^2} + \sigma^2 E(U^2 + V^2) \tag{37}$$

$$\frac{d^2F}{d\eta^2} - (\sigma^2 + 2iK^2)\,F = G\eta + R \tag{38}$$

where

$P_r' = \mu c/k_1$ is the Prandtl number in the porous medium. At the interface $P_r = P_r'$ because of $k = k_1$. The corresponding boundary conditions are

$$F = F_B \text{ and } \theta = \theta_B \text{ at } \eta = -1 \tag{39}$$

$$\frac{dF}{d\eta} = 0,\ G = 0 \text{ and } \theta = 0 \text{ at } \eta = -1 - \delta \tag{40}$$

where

$$\delta = \frac{\delta'}{L}$$

3.1. *Solution for the velocity field*

3.1.1. *Solution in chamber*

Keeping in view that the two boundary conditions (34) and (39) match at the interface $\eta = -1$, the eq. (31) can be solved by using (33), (34) and (39) to give

$$u + iv = \frac{R}{2iK^2}\left[\frac{\cosh(1+i)K\eta}{\cosh(1+i)K} - 1\right] + \frac{G}{2iK^2}\left[\frac{\sinh(1+i)K\eta}{\sinh(1+i)K} - \eta\right]$$

$$+ \frac{F_B}{2}\left[\frac{\cosh(1+i)K\eta}{\cosh(1+i)K} - \frac{\sinh(1+i)K\eta}{\sinh(1+i)K}\right] \tag{41}$$

where

$$F_B = U_B + iV_B$$

$$u_B = -\frac{(a_1d_1 - a_2d_2)}{(d_1^2 + d_2^2)\,K^2}\,G + \frac{(b_1d_1 - b_2d_2)}{(d_1^2 + d_2^2)\,K^2}\,R$$

$$v_B = -\frac{(a_1d_2 + a_2d_1)}{(d_1^2 + d_2^2)\,K^2}\,G + \frac{(b_1d_2 + b_2d_1)}{(d_1^2 + d_2^2)\,K^2}\,R$$

$$a_1 = \left[\frac{K\sinh 2K}{\cosh 2K - \cos 2K} - \frac{1}{2} - \frac{\alpha\sigma K^2(\sigma^2 - 2K^2)}{\sigma^4 + 4K^4}\right]$$

$$a_2 = \left[\frac{K\sin 2K}{\cosh 2K - \cos 2K} - \frac{1}{2} - \frac{\alpha\sigma K^2(\sigma^2 + 2K^2)}{\sigma^4 + 4K^4}\right]$$

$$b_1 = -\left[\frac{\sin 2K}{\cosh 2K + \cos 2K} + \frac{\alpha\sigma K(\sigma^2 - 2K^2)}{\sigma^4 + 4K^4}\right]$$

$$b_2 = \left[\frac{\sinh 2K}{\cosh 2K + \cos 2K} + \frac{\alpha\sigma K(\sigma^2 + 2K^2)}{\sigma^4 + 4K^4}\right]$$

$$d_1 = \left[\frac{2K\sinh 2K\cosh 2K}{\cosh^2 2K - \cos^2 2K} + \alpha\sigma\right]$$

$$d_2 = \left[\frac{2K\sin 2K\cos 2K}{\cosh^2 2K - \cos^2 2K} + \alpha\sigma\right].$$

3.1.2. *Solution in the porous medium*

From eq. (36), solutions for U and V are

$$U = -\frac{\sigma^2(G\eta + R)}{\sigma^4 + 4K^4} \tag{42}$$

$$V = \frac{2K^2(G\eta + R)}{\sigma^4 + 4K^4} \tag{43}$$

Here we note that, for $G \geqslant 0$ and $R < 0$, the filtration velocity U is always in the positive direction of x-axis while V is in the negative direction of Y-axis.

We know that the boundary condition (34) is based on the postulate that there exists a thin boundary layer adjacent to the interface. To obtain an expression for this boundary layer thickness δ, we solve eq. (38) using the boundary conditions (34), (39) and (40) and get

$$\delta = \text{Real part of}\left[\frac{1}{\sqrt{\sigma^2 + 2i\,K^2}}\tanh^{-1}\frac{\alpha\sigma}{\sqrt{\sigma^2 + 2i\,K^2}}\right] - 1. \tag{44}$$

From (44), it is obvious that the rotation has pronounced effects on the boundary layer thickness.

3.2. *Solution for the temperature distribution*

3.2.1. *Solution in chamber*

For simplicity, we assume that the viscous dissipation in the chamber is very much small in comparison to that in the porous bed. Hence we neglect the viscous dissipation term in the energy eq. (32) and the solution of (32) with the use of (33), (35) and (39) comes out to be

$$\theta = -\frac{P_r G}{4K^2}\left(\frac{R}{K^2} - v_B\right)\frac{(FX1 + FX2)}{(\cosh 2K + \cos 2K)}$$

$$-\frac{P_r G}{4K^2}\left(\frac{G}{K^2} - v_B\right)\frac{(FX1 - FX2)}{(\cosh 2K - \cos 2K)}$$

$$-\frac{u_B}{4K^2}\left[\frac{FY1 + FY2}{\cosh 2K + \cos 2K} - \frac{FY1 - FY2}{\cosh 2K - \cos 2K}\right] + C_1\eta + C_0 \quad (45)$$

where,

$P_r = P_r' = P$ at $\eta = -1$ and

$FX1 = \cosh K(1+\eta)\cos K(1-\eta)$

$FX2 = \cosh K(1-\eta)\cos K(1+\eta)$

$FY1 = \sinh K(1+\eta)\sin K(1-\eta)$

$FY2 = \sinh K(1-\eta)\sin K(1+\eta)$

$$C_1 = \tfrac{1}{2}(\theta_0 - \theta_B) + \frac{(P_r - P)G}{8K^2}\left(\frac{R}{K^2} - v_B\right) + \frac{(P_r + P)G}{8K^2}\left(\frac{G}{K^2} - v_B\right)$$

$$C_0 = \tfrac{1}{2}(\theta_0 + \theta_B) + \frac{(P_r + P)}{8K^2}\left(\frac{R}{K^2} - v_B\right) + \frac{(P_r - P)G}{8K^2}\left(\frac{G}{K^2} - v_B\right)$$

$$\theta_B = \frac{\theta_0}{(1 - 2N_u)} + \left(\frac{R}{K^2} - v_B\right)\frac{(P_r - P)G}{4K^2(1-2N_u)} + \left(\frac{G}{K^2} - v_B\right)\frac{(P_r+P)G}{4K^2(1-2N_u)}$$

$$-\left(\frac{R}{K^2} - v_B\right)\frac{(\sinh 2K - \sin 2K)PG}{2K(1-2N_u)(\cosh 2K + \cos 2K)}$$

$$+\left(\frac{G}{K^2} - v_B\right)\frac{(\sinh 2K + \sin 2K)PG}{2K(1-2N_u)(\cosh 2K - \cos 2K)}$$

3.2.2. *Solution in the porous medium*

Using the boundary conditions (39) and (40) eq. (37) can be solved to give

$$\frac{\theta}{P_r'} = a_4\eta^4 + a_3\eta^3 + a_2\eta^2 + a_1\eta + a_0 \tag{46}$$

where,

$$a_4 = -\frac{\sigma^2 EG^2}{12(\sigma^2+4K^2)}$$

$$a_3 = -\frac{\sigma^2 G(2RE+G)}{6(\sigma^2+4K^2)}$$

$$a_2 = -\frac{\sigma^2 R(RE+G)}{2(\sigma^2+4K^2)}$$

$$a_1 = \frac{\{(1+\delta)^4-1\}}{\delta}a_4 - \frac{\{(1+\delta)^3-1\}}{\delta}a_3 + \frac{\{(1+\delta)^2-1\}}{\delta}a_2 + \frac{\theta_B}{P\delta}$$

$$a_0 = \frac{(1+\delta)}{\delta}\left[\{(1+\delta)^3-1\}a_4 - \{(1+\delta)^2-1\}a_3 + \delta a_2 + \frac{\theta_B}{P}\right]$$

4. The Ekman-layer

When $K^2 \gg 1$, the highest order terms in (41) are multiplied by a small parameter $1/K^2$. Hence we can expect a boundary layer type flow at the two plates $\eta = \pm 1$.

For the boundary layer at the upper plate $\eta = 1$, we write $1 - \eta = \eta_1$, and obtain

$$u = -\frac{(R+G)}{2K^2}\exp(-K\eta_1)\sin K\eta_1 \tag{47}$$

$$v = -\frac{(R+G)}{2K^2}\{\exp(-K\eta_1)\cos K\eta_1 - 1\} - \frac{G}{2K^2}\eta_1 \tag{48}$$

This is the well known Ekman layer of the thickness of $O(1/K)$. It is interesting to note that it remains unaffected by the presence of permeable bed.

For the boundary layer at the lower plate $\eta = -1$, we write $1 + \eta = \eta_2$ and obtain

$$u = -\frac{(R+G)}{2K^2}\exp(-K\eta_2)\sin K\eta_2 + \exp(-K\eta_2)\{u_p\cos K\eta_2 + v_p\sin K\eta_2\} \tag{49}$$

$$v = -\frac{(R+G)}{2K^2}\{\exp(-K\eta_2)\cos K\eta_2 - 1\} - \frac{G}{2K^2}\eta_2$$

$$+ \exp(-K\eta_2)\{v_p \cos K\eta_2 - u_p \sin K\eta_2\} \tag{50}$$

where

$$u_p = -\frac{(1+\alpha\sigma)}{2K^2}(G+R)$$

$$v_p = \frac{(1+\alpha\sigma)}{2K^2}(G-R).$$

This may be identified as the modified Ekman layer as affected by the permeable bed. From (47) - (50) it can be seen that, for $R < 0$ and $G \leqslant 0$, in a certain core given by η_2, $\eta_1 \geqslant (1/K)$ about the axis of the channel the velocity in the direction of pressure gradient vanishes and the fluid will be moving in a direction transverse to the pressure — a result in keeping with the Taylor-Proudman theorem which states that all steady slow motions in a rotating inviscid fluid are necessarily two-dimensional.

5. Results

We have examined the effects of slip-velocities on the flow in chamber under the action of buoyancy force and permeability parameter. The profiles for slip-velocities U_B and V_B versus K have been displayed in figures 1 and 2 for different values of G and σ i.e. $G = 0$, 2, 6 and $\sigma = 5$, 10 and in all the calculations, R and α have been given a fixed value i.e. $R = -1$ and $\alpha = 0{\cdot}01$.

Consider first the profiles for the slip-velocity U_B in the vicinity of the porous bed. Figure 1 shows that the buoyancy force G has pronounced effects on U_B. When

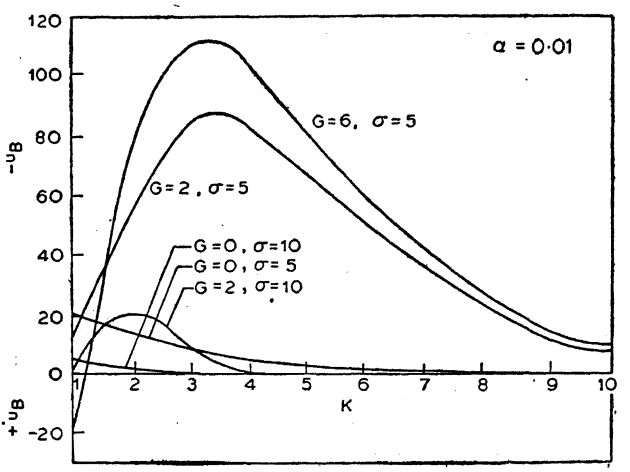

Figure 1. Profiles for u_B.

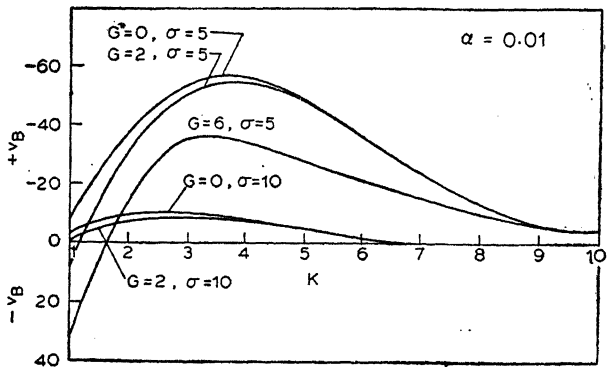

Figure 2. Profiles for v_B.

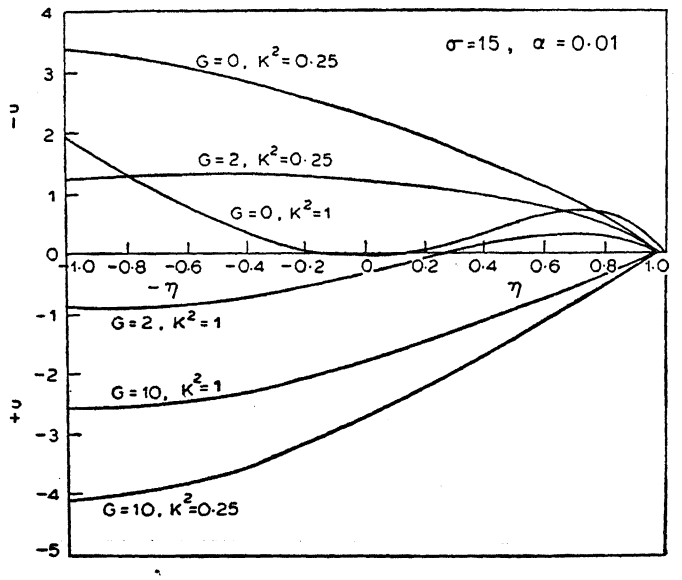

Figure 3. Velocity profiles.

σ is fixed and $G = 0$ (forced convection) the slip-velocity U_B decreases as the rotation K increases and when the rotation K is sufficiently large, the slip-velocity U_B, in the vicinity of the porous bed, is almost negligible. As G increases, U_B also increases abruptly producing humps but falls down to a constant value when the rotation becomes large. This reveals that U_B remains almost unaffected by G at large rotations and apart from this, the effect of G causes U_B to shift in the opposite direction at the initial rotation. When σ increases (i.e. the permeability decreases) U_B also decreases appreciably.

Now, consider the profiles for V_B in the y-direction as displayed in figure 2. It is found that the effect of G causes V_B to shift in the opposite direction but V_B remains almost unaffected by G at large rotations and V_B decreases as σ increases.

The velocity profiles u and v in chamber have been displayed in figures 3 and 4 for $K = 0{\cdot}5, 1$; $G = 0, 2, 10$ and $\sigma = 15$.

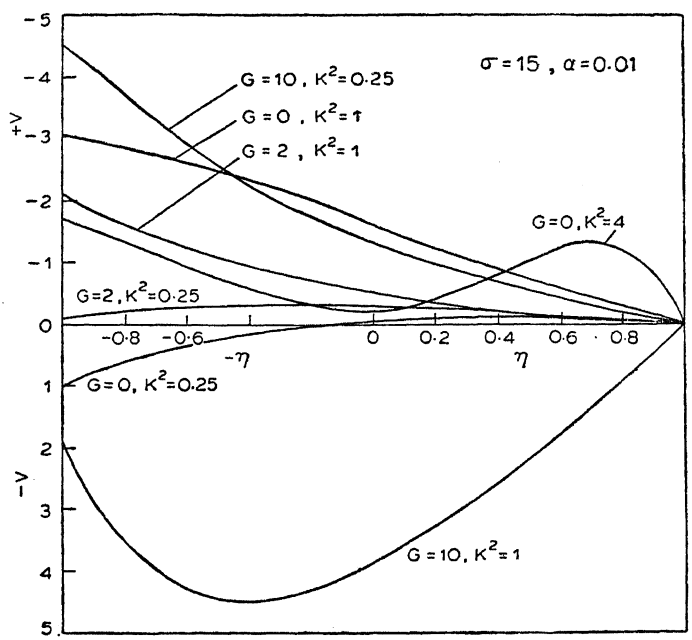

Figure 4. Velocity profiles.

Consider next the velocity profiles u for the primary flow in figure 3. It is seen that G has a significant effect on the flow. u decreases as G increases but, for a sufficiently large value of G, the fluid flows in the opposite direction. As we increase K and keep $G = 0$ (forced convection), the flow produces a hump near the solid wall and the fluid in the vicinity of the solid wall will be moving fastly. In fact, the velocity imparted to the fluid by the porous bed is quickly damped out and the fluid velocity in the central core in the primary flow direction almost becomes negligible. At a fixed rotation, the effect of G causes u to shift in the opposite direction. It is also found that the effect of rotation is also to decrease the velocity.

Now, consider the velocity profiles v for the secondary flow in figure 4. For small K, the fluid is moving partly in the positive and negative directions both when $G = 0$. As G increases the flow also increases in the positive direction. As we increase K, the effect of G causes the fluid to flow in the negative direction. On further increase of K, a hump is formed for $G = 0$ near the solid wall just like in the primary flow case.

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An integer arithmetic method to compute generalized matrix inverse and solve linear equations exactly

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Abstract. An algorithm that uses integer arithmetic is suggested. It transforms an $m \times n$ matrix to a diagonal form (of the structure of Smith Normal Form). Then it computes a reflexive generalized inverse of the matrix exactly and hence solves a system of linear equations error-free.

Keywords. Exact computation; integer arthematic; generalized matrix inverse; linear equations; Smith diagonai form.

1. Introduction

Whether a matrix A over a complex field is singular square, or rectangular, it has always a generalized inverse (g-inverse) over the (complex) field. The true inverse exists only when A is nonsingular (i.e., a square matrix whose determinant is not zero). However, a g-inverse of an $m \times n$ matrix of rank r involves considerable errors if the rth order submatrices are near-singular. Further, the rank shown by the g-inverse may be less than the actual rank. In fact, identical are the pitfalls when a (square) near-singular matrix is inverted.

We present here a method that uses integer arithmetic to

(i) transform an $m \times n$ integral matrix to a Smith Diagonal Form (defined later) without requiring to compute the greatest common divisors (GCDs) of the matrix elements as required in computing certain g-inverses (Hurt and Waid [4], Ben-Israel and Greville [2]),

(ii) compute a reflexive g-inverse (Bowman and Burdet [3], Ben-Israel and Greville [2], Krishnamurthy and Sen [5]),

(iii) obtain a solution vector x of $Ax=b$, b being 0 (null column vector) or not.

Since any computing system can represent only the rational numbers, we can, without any loss of generality, assume the inputs (here the matrix A and the right-hand-side column vector b) integral.

2. Definitions

2.1. *Integral vector and integral matrix*

Let (a) K = the ring of integers $0, \pm 1, \pm 2, \ldots$,
(b) K^m = the m dimensional vector space over K,

(c) $K^{m\times n}$ = the $m\times n$ matrices over K, and
(d) $K_r^{m\times n}$ = the $m\times n$ matrices with rank r over K.

Any element of K^m is an integral vector. Any element of $K^{m\times n}$ is an integral matrix, and any element of $K_r^{m\times n}$ is an integral matrix of rank r.

2.2. *Elementary row and column operations*

A sequence of elementary row and column operations used here consists of
(a) *Type* 1. Interchanging two rows (columns),
(b) *Type* 2. Subtracting an integral multiple of one row (column) from another row (column), and
(c) *Type* 3. Replacing a row (column) by an integral multiple of the row (column).

2.3. *Elementary integral matrix*

Any nonsingular matrix $P\in K^{m\times m}$ (any nonsingular matrix $Q\in K^{n\times n}$) which when pre- (post-) multiply a given matrix $A\in K_r^{m\times n}$, produces a combination of types 1, 2, and 3 operations.

2.4. *Equivalent matrices*

Two matrices A, $S\in K^{m\times n}$ are equivalent over K if there exist two elementary integral matrices $P\in K^{m\times m}$ and $Q\in K^{n\times n}$ such that $PAQ=S$.

2.5. *Smith diagonal form*

A matrix $S=(s_{ij})\in K_r^{m\times n}$ is the Smith Diagonal Form (SDF) of $A\in K_r^{m\times n}$ if

(a) $s_{ii}\neq 0$, $i=1(1)r$; (b) $s_{ij}=0$ otherwise, and (c) s_{ii} divides $s_{i+1,\,i+1}$, $i=1(1)r-1$.

2.6. *Generalized inverse of Smith diagonal form*

The g-inverse of SDF S is the matrix $S^+=(s_{ij}^+)\in K_r^{n\times m}$ if

(a) $s_{ii}^+=s_{ii}^{-1}$, $i=1(1)r$, and (b) $s_{ij}^+=0$ otherwise.

3. The method

Let $A\in K_r^{m\times n}$ do not have any zero row or zero column.

Step 1. *Computing* SDF $S=(s_{ij})\in K_r^{m\times n}$

(i) Make the elements at positions (1, 1), (2,2),..., nonzero using type 1 operations if any element at any of the positions (1, 1), (2, 2),..., is zero. Otherwise go to step 1(ii).
(ii) Multiply (type 3 operations) second, third,..., mth rows by $p_1, p_1p_2,\ldots, p_1p_2\cdots p_{m-1}$, respectively, where p_i is the element at position (i, i) or p_i is a number so that the

elementary matrices $P_1, P_2, \ldots$, where $P = \ldots P_2 P_1$, are integral (see step 2, and examples in section 6) or p_i is a number so that P is integral and the remainder in any division is zero.

(iii) Make zeros of all the elements below position (1, 1) in the first column using type 2 operations.

(iv) If the element at position (2, 2) is not zero then make zeros of all the elements below position (2, 2) in the second column using type 2 operations. Otherwise interchange $p_2 p_3 \ldots p_{s-1}$ times the second row and $1/(p_2 p_3 \ldots p_{s-1})$ times the sth row whose second element is nonzero (types 1 and 3 operations).

Make zeros of all the elements below position (2, 2) in the second column (if these are not all zero). Continue the process for the elements at positions (3, 3), (4, 4), etc.

Note. Steps 1(i)—1(iv) make the first r rows non-null with all the elements below positions (1, 1), (2, 2),..., etc. zero and other rows ($(r+1)$st to mth rows) null.

(v) Multiply second, third,..., rth columns by $q_1, q_1 q_2, \ldots, q_1 q_2 \ldots q_{r-1}$, respectively, where q_i is the element at position (i, i) or q_i is a number so that the elementary matrices $Q_1, Q_2, \ldots$, where $Q = Q_1 Q_2, \ldots$, are integral (see step 2 and examples in section 6) or q_i is a number so that Q is integral and the remainder in any division is zero. Multiply each of $(r+1)$st, $(r+2)$nd,..., nth columns by $q_1 q_2 \ldots q_{r-1}$.

(vi) Make zeros of all the elements after position (1, 1) in the first row using type 2 operations.

(vii) If the element at position (2, 2) is not zero then make zeros of all the elements after position (2, 2) in the second row using type 2 operations. Otherwise interchange $q_2 q_3 \ldots q_{s-1}$ times the second column and $1/(q_2 q_3 \ldots q_{s-1})$ times the sth column whose second element is nonzero (types 1 and 3 operations) and then make zeros of all the elements after position (2, 2) in second row (if these are not all zero). Continue the process for the elements at positions (3, 3), (4, 4), etc.

Note. Steps 1(i)–1(vii) give the SDF S. Also, in any divide operation denominator divides the numerator.

Step 2. *Computing* A^-.

(i) Compute the elementary matrix $P(Q)$ defined in sec. 2(iv), which is the product of all the elementary row (column) matrices, in the right order. Thus $PAQ = S$.

(ii) To obtain $A^- = QS^+ P$, compute

$$(i, j)\text{th element of } \alpha A^- = \sum_{k=1}^{r} q_{ik} p_{kj} (\alpha / s_{kk}),$$

$$i = 1\,(1)n, j = 1(1)m \text{ where } \alpha = s_{rr}.$$

Step 3. *Computing solution vectors.* To solve $Ax = b$, x being rational,

(i) compute $A \alpha A^- b$. If it is equal to αb, then solution exists. Otherwise, the system has no solution.

(ii) If $A \alpha A^- b = \alpha b$, then compute $\alpha A^- b$.

Obtain $\alpha x = \alpha A^- b + \alpha y - \alpha A^- A y$ for any $y \in K^n$.

Remarks

(i) *Computing g-inverse of a rational matrix.* Let B be an $m \times n$ nonintegral rational matrix. Then $A = \beta B$ is integral. Hence $B^- = \beta A^-$.
(ii) *Computing solution vector of a rational system.* Let $Cx = d$ be the nonintegral rational system. Then $Ex = f$, where $E = \gamma C$ and $f = \gamma d$, is an integral system.

4. Results

The method follows from the theorem and corollary below.

Theorem. Let $A \in K_r^{m \times n}$. Then A is equivalent over K to an SDF $S \in K_r^{m \times n}$. The proof follows from the construction of S described in step 1 of the method (Sec. 3).

Corollary. Let P and Q be elementary integral matrices and $PAQ = S$ be an SDF of $A \in K^{m \times n}$. Also, let $A^- = QS^+P$. Then

$$AA^-A = A, \; A^-AA^- = A^-.$$

Proof. $PAQ = S = SS^+S = PAQS^+PAQ = PAA^-AQ$. Hence $A = AA^-A$. A^-AA^- is proved similarly.

Note.

(i) *Integrality condition.* The A^- does not satisfy

$$A^-A \in K^{n \times n} \text{ and } AA^- \in K^{m \times m}$$

if (nontrivial) type 3 operations are used.
(ii) *Triangular matrices.* If type 1 operations are not needed then P and Q will be lower and upper triangular matrices, respectively. In such a case step 2(ii) can be written as

To obtain $A^- = QS^+P$, compute

$$(i, j)\text{th element of } \alpha A^- = \sum_{k=i}^{r} q_{ik} p_{kj} (\alpha / s_{kk}), \; i \geq j$$

$$= \sum_{k=j}^{r} q_{ik} p_{kj} (\alpha / s_{kk}), \; i < j$$

$$i = 1(1)r, \; j = 1(1)r \text{ where } \alpha = s_{rr}.$$

5. Use of modular arithmetic

For exact computation the modular arithmetic (Adegbeyeni and Krishnamurthy [1], Rao *et al* [6]) can be used only when the integer arithmetic (Sen and Shamim [7]),

in a general purpose computing system, demands too long precision operands. The modular arithmetic offers the parallelism in computation but the total computation is about u times the computation needed by the integer arithmetic, u being the number of prime bases used.

6. Examples

(1) $$A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 1 \\ 3 & 5 & 10 \end{bmatrix}$$

Here $p_1=2$, $p_2=6$, $p_3=10$.

Step 1(ii). Multiply second and third rows by 2 and 2×6, respectively (type 3):

$$P_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 3 & 5 \\ 8 & 12 & 2 \\ 36 & 60 & 120 \end{bmatrix}$$

Step 1(iii). To reduce the first column elements below position (1, 1) zero premultiply $P_1 A$ by P_2 (type 2):

$$P_2P_1A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -18 & 0 & 1 \end{bmatrix} \quad P_1A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & -18 \\ 0 & 6 & 30 \end{bmatrix}$$

Step 1(iv). The element at position (2, 2) is zero. So interchange p_2 times the second row and $(1/p_2)$ times the third row (here $s=3$) whose second element is nonzero (types 1 and 3):

$$P_3P_2P_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/6 \\ 0 & 6 & 0 \end{bmatrix} \quad P_2P_1A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & -108 \end{bmatrix}$$

Since the element below position (2, 2) in the second column is zero and $m=3$, we go to step 1(v).

Step 1(v). $q_1=2$, $q_2=1$, $q_3=-108$. Multiply second and third columns by 2, 2×1, respectively (type 3):

$$P_3P_2P_1AQ_1 = P_3P_2P_1A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 0 & 2 & 10 \\ 0 & 0 & -216 \end{bmatrix}$$

Step 1(vi). Make zeros of all the elements after position (1, 1) in the first row (type 2):

$$P_3P_2P_1AQ_1Q_2 = P_3P_2P_1AQ_1 = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 10 \\ 0 & 0 & -216 \end{bmatrix}$$

Step 1(vii). Since the element at position (2, 2) is not zero, make zeros of all the elements (here one) after position (2, 2) in the second row (type 2):

$$P_3P_2P_1AQ_1Q_2Q_3 = P_3P_2P_1AQ_1Q_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -216 \end{bmatrix} = S \text{ (SDF)}$$

$$P = P_3P_2P_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 2 \\ -24 & 12 & 0 \end{bmatrix}$$

$$Q = Q_1Q_2Q_3 = \begin{bmatrix} 1 & -3 & 10 \\ 0 & 2 & -10 \\ 0 & 0 & 2 \end{bmatrix}, \quad PAQ = S.$$

$$\alpha = s_{33} = -216$$

$$\alpha S^{+} = \begin{bmatrix} -108 & 0 & 0 \\ 0 & -108 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha A^{-1} = \alpha A^{-} = Q\alpha S^{+}P = \begin{bmatrix} -1320 & 120 & 648 \\ 888 & -120 & -432 \\ -48 & 24 & 0 \end{bmatrix}$$

(2) $$A = \begin{bmatrix} -1 & 2 & 3 & 3 \\ 2 & 5 & 6 & 3 \\ -5 & -8 & -9 & -3 \end{bmatrix}.$$

Multiply second and third rows by 1 and 1×5, respectively (type 3):

$$P_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 2 & 3 & 3 \\ 2 & 5 & 6 & 3 \\ -25 & -40 & -45 & -15 \end{bmatrix}$$

To reduce the first column elements below position (1, 1) zero pre-multiply P_1A by P_2 (type 2):

$$P_2P_1A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad P_1A = \begin{bmatrix} -1 & 2 & 3 & 3 \\ 0 & 9 & 12 & 9 \\ 0 & -90 & -120 & -90 \end{bmatrix}$$

To reduce the second column element below position (2, 2) zero pre-multiply P_2P_1A by P_3 (type 2):

$$P_3P_2P_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{bmatrix} \quad P_2P_1A = \begin{bmatrix} -1 & 2 & 3 & 3 \\ 0 & 9 & 12 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Multiply second, third, and fourth columns by 1, 1×9, 1×9, respectively (type 3):

$$P_3P_2P_1AQ_1 = P_3P_2P_1A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&9&0\\0&0&0&9\end{bmatrix} = \begin{bmatrix}-1&2&27&27\\0&9&108&81\\0&0&0&0\end{bmatrix}$$

To reduce the first row elements after position (1, 1) zero post-multiply $P_3P_2P_1AQ_1$ by Q_2 (type 2):

$$P_3P_2P_1AQ_1Q_2 = P_3P_2P_1AQ_1\begin{bmatrix}1&2&27&27\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = \begin{bmatrix}-1&0&0&0\\0&9&108&81\\0&0&0&0\end{bmatrix}$$

To reduce the second row elements after position (2, 2) zero post-multiply $P_3P_2P_1AQ_1Q_2$ by Q_3 (type 2):

$$P_3P_2P_1AQ_1Q_2Q_3 = P_3P_2P_1AQ_1Q_2\begin{bmatrix}1&0&0&0\\0&1&-12&-9\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}-1&0&0&0\\0&9&0&0\\0&0&0&0\end{bmatrix} = S$$

$$P = P_3P_2P_1 = \begin{bmatrix}1&0&0\\2&1&0\\-5&10&5\end{bmatrix},\quad Q = Q_1Q_2Q_3 = \begin{bmatrix}1&2&3&9\\0&1&-12&-9\\0&0&9&0\\0&0&0&9\end{bmatrix}$$

$$a = s_{22} = 9,\quad aS^+ = \begin{bmatrix}-9&0&0\\0&1&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$aA^- = QaS^+P = \begin{bmatrix}-5&2&0\\2&1&0\\0&0&0\\0&0&0\end{bmatrix}$$

Note that $PAQ=S$, and P and Q are lower and upper triangular matrices, respectively since type 1 operations are not used.

$$AaA^- = \begin{bmatrix}9&0&0\\0&9&0\\9&-18&0\end{bmatrix}$$

If $b=(7\ \ 16\ \ -25)^t$ then $A\alpha A^{-}b=\alpha b$ and hence solutions exist. If $b=(8\ \ 16\ \ -25)^t$ then $A\alpha A^{-}b\neq\alpha b$ and hence solutions do not exist. For $b=(7\ \ 16\ \ -25)^t$ the solution vector $\alpha x=(-3\ \ 30\ \ 0\ \ 0)^t$.

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On a class of nonlinear integral equations

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Abstract. In this paper we consider abstract equations of the type $K_v v+v=w_0$, in a closed convex subset of a separable Hilbert space H. For each v in the closed convex subset, $K_v : H \to H$ is a bounded linear map. As an application of our abstract result we obtain an existence result for nonlinear integral equations of the type $v(s)+v(s)\int_0^1 k(s,t)\,v(t)\,dt=W_0(s)$ in the space $L^2[0,1]$.

Keywords. Existence theorem; Hammerstein equations; Chandrasekar's H-equation.

1. Introduction

Let X be a Banach space with dual X^*. Abstract equations of the type $KNv+v=0$ have been considered by many authors and have a vast literature (see [1]). Here $K: X \to X^*$ is a linear map and $N: X^* \to X$ a nonlinear map. Equations of this type are called Hammerstein type equations. Recently equations of the type $K(v)\,Nv+v=0$, called 'Generalized Hammerstein type equations', have been considered by Petry [6, 7] and by the authors in [10]. Here for each $v \in X$, $K(v): X^* \to X$ is a linear map and $N: X \to X^*$ a nonlinear map. Stuart [8] while dealing with integral equations of the type

$$v(s) + v(s) \int_0^1 k(s, t)\, v(t)\, dt = w_0(s) \tag{1}$$

has expressed the fear that these equations may not fall under the category of Hammerstein type equations. But we observe that (1) is a particular case of the generalized Hammerstein equation. For, if we define for each v in L^2, a linear operator $K_v: L^2 \to L^2$ by

$$(K_v u)(s) = u(s) \int_0^1 k(s, t)\, v(t)\, dt,$$

($|K(s, t)|$ is assumed to be essentially bounded on $[0, 1] \times [0, 1]$) then solving equation of the type (1) reduces to solving abstract equation of the type $K_v v + v = w_0$ in L^2.

In this paper we obtain existence results for equations of the type $K_v v+v=w_0$ and as an application obtain existence results for equations of type (1) which leads to a solution of the equation

$$1 + v(s) \int_0^1 \frac{s}{s+t}\, v(t)\, x(t)\, dt = v(t).$$

The above equation is Chandrasekar's H-equation (see [2]). Our existence results are very similar to those of Stuart [9], but our approach is different.

In the next section we recall certain definitions and results from fixed point theory of multivalued mappings in topological vector spaces.

2. Some definitions and results from fixed point theory

Definition 1. If X and Y are topological spaces, T a mapping of X into 2^Y then T is said to be upper semicontinuous if for each point x_0 of X and an arbitrary neighbourhood V of $T(x_0)$ in Y, there exists a neighbourhood U of x_0 in X such that for all x in U, $T(x) \subset V$.

Lemma 1. Let T be a mapping of a compact space K into 2^{K_1} with $T(x)$ closed for each x in K. Then T is upper semicontinuous from K to 2^{K_1} if and only if its graph

$$G(T) = \{[x, y] : x \in K,\ y \in K_1,\ y \in T(x)\}$$

is closed in $K \times K_1$.

Theorem 1 (Himmelberg). Let S be a nonvoid convex subset of a separated locally convex space E. Let $f: S \to S$ be an U.S.C. multifunction such that $f(x)$ is closed and convex for each $x \in S$ and $f(S)$ is contained in some compact set K of S. Then f has a fixed point.

The above definitions and results are taken from [2] and [4]. Throughout this paper we use ' $\to$ ' to denote strong convergence and ' $\rightharpoonup$ ' to denote weak convergence. $B[0, r]$ denotes the closed ball with radius ' r '.

3. Main results

Theorem 2. Let X be a separable reflexive Banach space and A a closed convex subset of X. Assume that for each $v \in A$, we have a map $\varphi_v : X \to Y$, Y a topological vector space, such that the following hold:

(i) $v_n \rightharpoonup v$ in A and $u_n \rightharpoonup u$ in X implies that $\varphi_{u_n}(v_n) \rightharpoonup \varphi_u(v)$.

(ii) For each v in A, the set $A_v = \{u \in X : \varphi_v(u) = 0\}$ is a nonvoid, convex set and intersects $A \cap B\ [0, r]$ for some fixed ' r ' in a nonvoid set.

Then the equation $\varphi_v(v) = 0$ has a solution in A.

Proof: Define a map T: $A \cap B\ [0, r] \to 2^{A \cap B\ [0, r]}$ by

$$Tv = \{u \in A \cap B\ [0, r] : \varphi_v(u) = 0\} \text{ for each } v.$$

Obviously for each v in $A \cap B\ [0, r]$, Tv is a nonvoid convex set [assumption (i)]. Further for each v, Tv is a closed set in the weak topology of X. For if $u_n \rightharpoonup u$

in Tv, then $\varphi_v(u_n) \rightharpoonup \varphi_v(u)$ implies $\varphi_v(u) = 0$ since each $\varphi_v(u_n) = 0$. Hence $u \in Tv$. We now show that the above map T is U.S.C. when $A \cap B[0,r]$ is equipped with the weak topology. To prove this we use lemma 1 and the fact that a weakly closed and bounded set in a reflexive space is weakly compact. In the notation of the lemma, suppose $(x_n, y_n) \in G(T)$ and $(x_n, y_n) \rightharpoonup (x, y)$. Then since $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, we have from (i) that $\varphi_{x_n}(y_n) \rightharpoonup \varphi_x(y)$. Since $y_n \in Tx_n$, $\varphi_{x_n}(y_n) = 0$ and hence $\varphi_x(y) = 0$ or $y \in Tx$. Therefore $G(T)$ is weakly closed. So by lemma 1, T is U.S.C. Since T is an U.S.C. map mapping the compact set $A \cap B\ [0, r]$ into $2^{A \cap B\,[0,r]}$ we have by theorem 1 that T has a fixed point, i.e. there exists v such that $v \in Tv$, which implies $\varphi_v(v) = 0$. Hence the result.

Remark 1. If for each v in the above theorem, $\varphi_v(u) = K_v(u) + u - w_0$, then we obtain an existence result for equation of the type $K_v v + v = w_0$, as we see in the next theorem.

In the following theorem we obtain an existence result for equations of the type

$$v(s) + v(s) \int_0^1 k(s, t) v(t) dt = w_0(s) \qquad (*)$$

in the space of square integrable functions $L^2\ [0,1\ ]$.

Theorem 3. Suppose that

(i) $k(s, t) \geqslant 0$ a.e. on $[0, 1] \times [0, 1]$;

(ii) ess sup $\int_0^1 k^2(s, t)\, dt < \infty$;

(iii) $w_0(s) \geqslant 0$ a.e. on $[0, 1]$.

Then (1) has a solution $v \in L^2\ [0, 1]$ such that

$$0 \leqslant v(s) \leqslant w_0(s) \text{ a.e. on } [0, 1].$$

Proof: Let $A = \{v \in L^2 : v(s) \geqslant 0 \text{ a.e. on } [0, 1]\}$ and let

$$(K_v u)(s) = u(s) \int_0^1 k(s, t)\, v(t) dt.$$

Then for each $v \in A$, K_v is a bounded linear operator of L^2 into itself by (ii).

We now prove that if $v_n \rightharpoonup v$ in A and $u_n \rightharpoonup u$ in L^2, then $K_{v_n}(u_n) \rightharpoonup K_v(u)$ in $L^2\ [0, 1]$. First we note that if $v_n \rightharpoonup v$ in A then $K_{v_n}(u) \to K_v(u)$. For

$$K_{v_n}(u)(s) = u(s) \int_0^1 k(s, t)\, v_n(t)\, dt$$

$$= \int_0^1 K_u(s, t)\, v_n(t) dt,$$

where $K_u(s,t) = u(s)\, k(s,t)$. Since $K_u(s,t)$ is a L^2 Kernel, we have the corresponding integral operator completely continuous and hence $K_{v_n}(u) \to K_v(u)$. Consider

$$\langle K_{v_n}(u_n) - K_v(u), w \rangle \text{ for some arbitrary } w \in L^2;$$

$$\langle K_{v_n}(u_n) - K_v(u), w\rangle = \langle K_{v_n}(u_n) - K_v(u_n), w\rangle + \langle K_v(u_n) - K_v(u), w\rangle$$

$$= \int_0^1 w(s)\, u_n(s) \left(\int_0^1 k(s,t)\,(v_n(t) - v(t))\, dt\right) ds$$

$$+ \int_0^1 w(s)\,(u_n(s) - u(s)) \left(\int_0^1 k(s,t)\, v(t)\, dt\right) ds$$

$$= \int_0^1 u_n(s) \left(\int_0^1 k_w(s,t)\,(v_n(t) - v(t))\, dt\right) ds$$

$$+ \int_0^1 (u_n(s) - u(s) \left(\int_0^1 k_w(s,t)\, v(t)\, dt\right) ds.$$

Since $\int_0^1 k_w(s,t)\, v_n(t)\, dt \to \int_0^1 k_w(s,t)\, v(t)\, dt$ and (u_n) is a bounded sequence, we have that

$$\int_0^1 u_n(s) \left(\int_0^1 k_w(s,t)\,(v_n(t) - v(t))\, dt\right) ds \to 0.$$

Also since $\int_0^1 k_w(s,t)\, v(t)\, dt \in L^2$ and $u_n \rightharpoonup u$, we have

$$\int_0^1 (u_n(s) - u(s)) \left(\int_0^1 k_w(s,t)\, v(t)\, dt\right) ds \to 0.$$

Hence $\langle K_{v_n}(u_n) - K_v(u), w\rangle \to 0$. Since $w \in L^2$ is arbitrary, we have $K_{v_n}(u_n) \rightharpoonup K_v(u)$.
Now set $\varphi_v(u) = K_v(u) + u - w_0$ for $v \in A$.
For v in A, $\int_0^1 k(s,t)\, v(t)\, dt \geqslant 0$ and so

$$u(s) = -\frac{w_0(s)}{1 + \int_0^1 k(s,t)\, v(t)\, dt}$$

is the only solution of $\varphi_v(u) = 0$ and $0 \leqslant u(s) \leqslant w_0(s)$. By setting $r = \| W_0 \|$; we see that the hypotheses of theorem 3 are satisfied. This completes the proof.

We next consider a particular case of $k(s,t)$ and obtain as a corollory, a result similar to theorem 2 of [8].

Corollory 1: Suppose

(i) $k(s,t) = \dfrac{R(s,t)}{s^2 - t}$ for $s \neq t$.

(ii) $R(s,t) < 0$ if $s < t$

$R(s,t) < 0$ if $s > t$.

(iii) There exists a constant $\gamma > 1$ such that

$$|R(s,t)| < M\,|s - t|^{\gamma}\,|s + t|.$$

Then the equation

$$v(s) + v(s) \int_0^1 [R(s, t)/(s^2 - t^2)]\, v(t)\, dt = 1$$

has a solution in L^2 [0, 1]. Also $0 \leqslant v(s) \leqslant 1$ a.e.

Remark 2. Equations of type (1) have been tackled by many including Stuart [9] and Leggett [5]. In fact Leggett has obtained slightly more general results. But our proof here is new and also the joint weak continuity proved in theorem 3, leads to a simple existence theorem for the Chandrasekar's H-equation, as shown in the next section.

Chandrasekar's H-equation

$$1 + v(s) \int_0^1 \frac{s}{s+t} v(t)\, \chi(t)\, dt = v(t) \tag{2}$$

occupies an important place in the theory of radiative transfer in semi-infinite atmospheres [2]. The known function is usually assumed to be non-negative, bounded and measurable on [0, 1] and henceforth we shall require this. We note that $s/s+t$ is bounded and continuous on $[0, 1] \times [0, 1] - \{(0, 0)\}$.

To begin with, we state two lemmas without proof which are based on similar manipulations as in [2] and [9].

Lemma 2: Suppose that $v \in L^2$ [0, 1] is a solution of (2). Then $\int_0^1 \chi(t) dt \leqslant 1/2$ and $v(s) \geqslant 1$ a.e. on [0, 1].

The next result enables us to replace eq. (2) by a suitable equation, which will lead to a solution of (2), by joint weak continuity.

Lemma 3: Suppose that $\int_0^1 \chi(t) dt \leqslant 1/2$ and that $v \in L^2$ [0, 1] is a positive solution of the equation

$$v(s) \left\{1 - 2 \int_0^1 \chi(t) dt\right\}^{1/2} + v(s) \int_0^1 [t/(s+t)]\, \chi(t)\, v(t)\, dt = 1. \tag{3}$$

Then

$$\int_0^1 \chi(s)\, v(s)\, ds = 1 - \left\{1 - 2 \int_0^1 \chi(s)\, ds\right\}^{1/2}$$

and v is a solution of eq. (2).

Theorem 4: Equation (2) has a solution in L^2 [0, 1] if and only if

$$\int_0^1 \chi(s)\, ds \leqslant 1/2.$$

All the solution of (2) are greater than or equal to unity a.e. on [0, 1].

Proof: According to the preceding results, it is sufficient to establish the existence of a positive solution of eq. (3) in L^2 [0, 1], when

$$\int_0^1 \chi(s)\, ds \leqslant 1/2.$$

For $\int_0^1 \chi(s)\,ds < 1/2$, the result follows immediately from theorem 3.

Suppose now that $\int_0^1 \chi(s)\,ds = 1/2$.

Let $\chi(s, n) = (1 - 1/n)\,\chi(s)$, n a positive integer, and $K(s, t, n) = [t/(s+t)]\,\chi(s, n)$ for $s, t \in [0, 1]$, $(s, t) \neq (0, 0)$. Consider the equation

$$1 = u(s)\left\{1-2\int_0^1 \chi(s, n)\,ds\right\}^{1/2} + u(s)\int_0^1 k(s, t, n)\,u(t)\,dt \tag{4}$$

$$= u(s)\frac{1}{n^{1/2}} + u(s)\int_0^1 k(s, t, n)\,u(t)\,dt. \tag{5}$$

If $u \in L^2[0, 1]$ is a positive solution of (4) for some n, we have from lemma 2,

$$\int_0^1 \chi(s, n)\,u(s)\,dt = 1 - \left\{1 - 2\int_0^1 \chi(s, n)\,ds\right\}^{1/2} = 1 - \frac{1}{n^{1/2}}. \tag{6}$$

But from (5), it follows that

$$1/u(s) \leqslant \frac{1}{n^{1/2}} + \int_0^1 \chi(t, n)\,u(t)\,dt \text{ for } s \text{ in } [0, 1].$$

This implies $1/u(s) \leqslant 1$ a.e. on $[0, 1]$ due to (6).

Using the above fact, it follows from (5), that

$$1/u(s) \geqslant \int_0^1 [t/(s+t)]\,\chi(t)\,dt.$$

Let u_n denote a solution of (4), then we have

$$1 = u_n(s)\left\{1-2\int_0^1 \chi(s, n)\,ds\right\}^{1/2} + u_n(s)\int_0^1 k(s, t, n)\,u_n(t)\,dt.$$

i.e. $$1 = u_n(s)\left\{1-2\left(\int_0^1 \chi(s)\,ds - \int_0^1 \chi((s)/n)\,ds\right)\right\}^{1/2}$$
$$+ u_n(s)\int_0^1 (t/s+t)\,\chi(t)\,u_n(t)\,dt$$
$$- (u_n(s)/n)\int_0^1 (t/s+t)\,\chi(t)\,u_n(t)\,dt.$$

i.e. $$1 = \frac{u_n(s)}{n^{1/2}} + u_n(s)\int_0^1 (t/s+t)\,\chi(t)u_n(t)\,dt$$
$$- 1/n\,u_n(s)\int_0^1 (t/s+t)\,\chi(t)\,u_n(t)\,dt \tag{7}$$

Since $\{u_n\}$ is a bounded sequence in L^2, it has a weakly convergent subsequence, which we shall not distinguish. Let $u_n \rightharpoonup u_0$, then from (7) and the joint weak continuity established in theorem 3, it follows that,

$$1 = u_0(s)\int_0^1 (t/s+t)\,\chi(t)\,u_0(t)\,dt.$$

This completes the proof.

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Large and high subgroup

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Abstract. It is proved that a high subgroup of a large subgroup is closed in a high subgroup of the group itself. Also a necessary and sufficient condition for a subgroup to be a pureabsolute summand is obtained.

Keywords. Subgroup

1. Introduction

The concept of large subgroups was first introduced by Pierce [7]. Since then a number of papers by Meggiben [6], and Abdullah *et al* [1] have been published on the various properties and representations of large subgroups. Abdulla *et al* [1] proved that large subgroups are relatively closed. We consider the question: What are the subgroups whose relative closures are large? This question is partially answered in § 2 and it is proved that a high subgroup of a large subgroup is relatively closed in a high subgroup of the group itself. As an application, some results of Σ-groups are also deduced. In § 3 a necessary and sufficient condition for a subgroup to be a pure-absolute summand is obtained. All groups considered here are abelian primary groups. All topological references are of p-adic topology, and the notations are those of Fuchs [3].

2. Relative closure

To start with we recall some basic definitions. A subgroup L of a p-group G is called large if $G=L+B$ for every basic subgroup B of G. A group G is called a Σ-group if all its high subgroups are direct sums of cyclic groups. A subgroup X of G is relatively closed under the p-adic topology if and only if $(G/X)^1=0$. A subgroup H of G is called the relative closure of X in G if and only if $H/X=(G/X)^1$.

Now we state a result whose proof is immediate.

Proposition 2.1

If G is a torsion group and $G = \oplus\, \Sigma\, G_\alpha$ then $G^1 = \oplus\, \Sigma G_\alpha{}^1$.

Proposition 2.2

If L is a large subgroup of G then L is the relative closure of every subgroup H of L such that L/H is divisible.

Proof: Let L/H be divisible for some subgroup H of L, then $G/H=R/H \oplus L/H$. Now appealing to | Cor. 1, 1 | we get $(R/H)^1=0$ and hence by proposition 2.1 $(G/H)^1 = L/H$ and the result follows.

Corollary 2.1

If L is a large subgroup of G then L is the relative closure of $L \cap B$ for every basic subgroup B of G.

This immediately follows from | lemma 2.12; 7 | and proposition 2.2.

Now we have the following theorem which proves that a high subgroup of a large subgroup is relatively closed in a high subgroup.

Theorem 2.1

Let G be a group with $G^1 \neq 0$ and L be a large subgroup of G, then a high subgroup H of L is relatively closed in a high subgroup of G.

Proof: Let H be a high subgroup of L then L/H is divisible | theorem 5; 4 | and we have $G/H=L/H \oplus A/H$ for some subgroup A of G. Now appealing to | lemma 2.4; 1 | it is easy to see that $A \cap G^1=0$. Now for proving A to be high subgroup of G, it is sufficient to show that $(A, x) \cap G^1 \neq 0$ for every $x \in L$, $x \notin A$. If $(A, x) \cap G^1 = 0$ then $(H, x) \cap L^1 = 0$, a contradiction. Hence A is a high subgroup of G. Now it is easy to see that $(A/H)^1 = L/H$. Hence the assertion.

The following corollary is immediate:

Corollary 2.2

If L is a large subgroup of G and H is a high subgroup of L such that H is countable then H is a relatively closed in a maximal basic subgroup of G.

Corollary 2.3

If L is a large subgroup of G and L contains a countable high subgroup, then L and G are Σ-groups.

Proof: If H is a countable high subgroup of L then clearly H is a maximal basic subgroup of L and hence by | theorem 9; 5 | L is a Σ-group. Appealing to corollary 2.2, we get G to be a Σ-group.

Theorem 2.2

If G is a group with $G^1 \neq 0$ then G is a Σ-group if and only if given a large subgroup L of G, L is Σ-group.

Proof: If G is a Σ-group and L is large in G then $L^1 = L \cap G^1$ and by | theorem 11; 5 | L is a Σ-group. Conversely if L is a Σ-group and H is a high subgroup of L then H is direct sum of cyclic groups and $G/H = L/H \oplus A/H$ where A (as done in theorem 2.1) is a high subgroup of G. Now appealing to | theorem 4.7; 2 | we get A to be maximal basic subgroup of G. Hence G is a Σ-group.

Corollary 2.4

If G is a group with $G^1 \neq 0$ then G is a Σ-group if and only if p^nG for any n is a Σ-group.

3. Pure-absolute summand

A subgroup H of G is called pure-N-high if it is maximal disjoint with N among the pure subgroups. A subgroup H is called pure-absolute summand if for every pure H-high subgroup A, $G=A \oplus H$.

Theorem 3.1

A subgroup A of G is pure-absolute summand if and only if N is pure subgroup and $A \cap N=0$ implies $A+N$ is pure in G.

Proof: Let A be pure-absolute summand of G then we can find a pure$A$-high subgroup B such that $N \subseteq B$ and $G=A \oplus B$. Now $(A+N) \cap p^nG=(A+N) \cap (p^nA+p^nB)$ $=p^nA+(A+N) \cap P^nB$. It is easy to see that $(A+N) \cap p^nB=p^nN$. Hence $A+N$ is pure subgroup of G. Conversely let B be a pure A-high subgroup of G, then appealing to [theorem 14; 2] B is A-high. Hence $A+B$ is an essential subgroup of G and we get $(A+B)[p]=G[p]$. By assumption $A+B$ is pure, consequently $G=A$ $\oplus B$ and A is pure-absolute summand of G.

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A cylindrically symmetric universe in general relativity

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Abstract. A cylindrically symmetric universe with two degrees of freedom which is of Petrov Type I degenerate, has been derived. Various physical and geometrical properties of the model have also been discussed.

Keywords. Cylindrical cosmological model; general relativity.

1. Introduction

In recent years, Ellis and Maccullum [2] have studied a class of homogeneous models representing perfect fluid distributions. Heckmann and Schucking [3] have derived more general models representing incoherent matter distribution. In our previous paper [5] we have derived a gravitationally non-degenerate cosmological model of plane symmetry in general relativity. In this paper, we construct a model of cylindrically symmetric space time with two degrees of freedom which is of Petrov Type I degenerate, the degeneracy being in the y and z directions. The energy momentum tensor has been assumed to be that of perfect fluid. The reality conditions impose restrictions on the time during which the model exists. The expressions for expansion, rotation, shear, path of test particle and the generalized Doppler effect have been obtained.

We consider the metric in the form

$$dS^2 = A^2 (dx^2 - dt^2) + C^2 dz^2 + (B^2 + D^2) dy^2 + 2CDdydz \tag{1}$$

where the metric potentials are functions of time alone. The energy momentum tensor for perfect fluid distribution is given by

$$T_i^j = (\epsilon + p) v_i v^j + p \delta_i^j \tag{2}$$

together with

$$g_{ij} v^i v^j = -1 \tag{3}$$

p being the pressure, ϵ the density and v^i the flow vector satisfying (3). We assume the co-ordinates to be co-moving so that $v^1 = v^2 = v^3 = 0$ and $v^4 = 1/A$.

The field equations

$$-8\pi T_i^j = R_i^j - \tfrac{1}{2} R g_i^j + \Lambda \delta_i^j \tag{4}$$

for the metric (1) are

$$8\pi p = \frac{1}{A^2}\left\{\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} - \frac{B_4C_4}{BC} - \frac{B_{44}}{B} - \frac{C_{44}}{C}\right\} - \frac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 - \Lambda, \tag{5}$$

$$8\pi p = -\frac{1}{A^2}\left\{\frac{A_{44}}{A} + \frac{B_{44}}{B} - \frac{A_4^2}{A^2}\right\} - \frac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_{44}$$

$$+ \frac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B} - \frac{3C_4}{C}\right) - \frac{3}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 - \Lambda, \tag{6}$$

$$8\pi p = -\frac{1}{A^2}\left\{\frac{A_{44}}{A} + \frac{C_{44}}{C} - \frac{A_4^2}{A^2}\right\} + \frac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_{44}$$

$$- \frac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B} - \frac{3C_4}{C}\right) + \frac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 - \Lambda, \tag{7}$$

$$8\pi\epsilon = \frac{1}{A^2}\left\{\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC}\right\} - \frac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 + \Lambda \tag{8}$$

and

$$\left(\frac{D}{C}\right)_{44} - \left(\frac{D}{C}\right)_4\left\{\frac{B_4}{B} - \frac{3C_4}{C}\right\} = 0 \tag{9}$$

Choosing the tetrad $\lambda_i^{\ j}$ as

$$\lambda_i^{\ j} = \begin{bmatrix} \frac{1}{A} & 0 & 0 & 0 \\ 0 & \frac{1}{C} & 0 & 0 \\ 0 & -\frac{D}{BC} & \frac{1}{B} & 0 \\ 0 & 0 & 0 & \frac{1}{A} \end{bmatrix}, \tag{10}$$

the non-vanishing physical components of conformal curvature tensor are

$$C_{(1212)} = -C_{(3434)}$$

$$= \frac{1}{6A^2}\left[\frac{A_{44}}{A} + \frac{C_{44}}{C} - \frac{2B_{44}}{B} - \frac{A_4^2}{A^2} + \frac{B_4C_4}{BC} + \frac{3A_4B_4}{AB}\right.$$

$$\left. - \frac{3A_4C_4}{AC} - \frac{2C^2}{B^2}\left\{\left(\frac{D}{C}\right)_4\right\}^2\right], \tag{11}$$

$$C_{(1313)} = -C_{(2424)}$$

$$= -\frac{1}{6A^2}\left[\frac{A_{44}}{A}+\frac{B_{44}}{B}-\frac{2C_{44}}{C}-\frac{A_4^2}{A^2}+\frac{B_4C_4}{BC}\right.$$

$$\left.+\frac{3A_4C_4}{AC}-\frac{3A_4B_4}{AB}+\frac{C^2}{B^2}\left\{\left(\frac{D}{C}\right)_4\right\}^2\right], \tag{12}$$

$$C_{(1414)} = -C_{(2323)}$$

$$= -\frac{1}{6A^2}\left[\frac{B_{44}}{B}+\frac{C_{44}}{C}-\frac{2A_{44}}{A}+\frac{2A_4^2}{A^2}-\frac{2B_4C_4}{BC}+\frac{C^2}{B^2}\left\{\left(\frac{D}{C}\right)_4\right\}^2\right], \tag{13}$$

$$C_{(1213)} = C_{(2434)}$$

$$= \frac{C}{4BA^2}\left[\left(\frac{D}{C}\right)_{44}+\left(\frac{D}{C}\right)_4\left\{\frac{3C_4}{C}-\frac{2A_4}{A}-\frac{B_4}{B}\right\}\right]. \tag{14}$$

The suffix 4 after the symbols A, B, C and D indicates ordinary differentiation with respect to t.

2. Solution of the field equations

Equations (5)–(9) are five equations in six unknowns A, B, C, D, ϵ and p. For the complete determination of this set, we need an extra condition. We assume $C_{(1212)} = C_{(1313)}$. The gravitational field will obviously be of degenerate Petrov Type I (Petrov [4]). Therefore from eqs (11) and (12), we have

$$\frac{C_{44}}{C}-\frac{B_{44}}{B}+\frac{2A_4}{A}\left(\frac{B_4}{B}-\frac{C_4}{C}\right)=\left[\frac{CD_4-DC_4}{BC}\right]^2. \tag{15}$$

Equation (9) on integration gives

$$\frac{CD_4-DC_4}{BC}=\frac{K}{C^2} \tag{16}$$

where K is a constant of integration. From eqs (5), (6) and (9), we have

$$\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}-\frac{B_4C_4}{BC}-\frac{C_{44}}{C}+\left(\frac{A_4}{A}\right)_4=-\frac{1}{2}\left[\frac{CD_4-DC_4}{BC}\right]^2. \tag{17}$$

Also from eqs (6), (7) and (9), we have

$$\frac{C_{44}}{C}-\frac{B_{44}}{B}=\left[\frac{CD_4-DC_4}{BC}\right]^2. \tag{18}$$

Equations (16) and (18) lead to

$$\frac{C_{44}}{C} - \frac{B_{44}}{B} = \frac{K^2}{C^4} \tag{19}$$

From eqs (16) and (17), we have

$$\frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} - \frac{B_4 C_4}{BC} - \frac{C_{44}}{C} + \left(\frac{A_4}{A}\right)_4 = -\frac{K^2}{2C^4}. \tag{20}$$

From (15) and (18), we have

$$\frac{A_4}{A}\left(\frac{B_4}{B} - \frac{C_4}{C}\right) = 0. \tag{21}$$

Since $B \neq C$, eq. (21) on integration gives

$$A = \text{constant} = a \text{ (say)}. \tag{22}$$

From (20) and (22), we have

$$\frac{B_4 C_4}{BC} + \frac{C_{44}}{C} = \frac{K^2}{2C^4}. \tag{23}$$

Equations (19) and (23) lead to

$$B = \frac{at + b}{C} \tag{24}$$

where a and b are constants of integration. Equations (19) and (24) lead to

$$\left\{(at + b)\frac{C_4}{C}\right\}_4 = \frac{K^2(at + b)}{2C^4} \tag{25}$$

which on integration gives

$$C^2 = \frac{2K(at + b)}{N} \cosh\left\{\frac{N}{2a}(la - \log(at + b)\right\} \tag{26}$$

l and N being constants of integration. From eqs (24) and (26), we have

$$B^2 = \frac{N(at + b)}{2K} \operatorname{sech}\left\{\frac{N}{2a}(la - \log(at + b))\right\} \tag{27}$$

From eqs (19), (24) and (26), we have

$$D = C\left[L - \frac{N}{2K}\tanh\left\{\frac{N}{2a}(la - \log(at + b))\right\}\right] \tag{28}$$

L being a constant of integration.

After suitable transformation of co-ordinates, the metric of space-time can be put into the form

$$dS^2 = \frac{\alpha^2}{a^2}(dX^2 - dT^2) + T\cosh\phi\, dZ^2$$

$$+ T\left[(\beta^2 + 4\gamma^2)\cosh\phi - 4\beta\gamma\sinh\phi\right] dY^2$$

$$+ 2T\left[2\gamma\cosh\phi - \beta\sinh\phi\right] dY\, dZ \tag{29}$$

α, β and γ being arbitrary constants of integration and

$$\phi = \frac{N}{2a}(la - \log T).$$

3. Some physical and geometrical features

The pressure and density for the model (29) are given by

$$8\pi p = \frac{1}{\alpha^2 a^2}\left[\frac{a^2}{4T^2} - \frac{\beta^2}{16T^2}\right] - \Lambda \tag{30}$$

and

$$8\pi\epsilon = \frac{1}{\alpha^2 a^2}\left[\frac{a^2}{4T^2} - \frac{\beta^2}{16T^2}\right] + \Lambda. \tag{31}$$

The model has to satisfy the reality conditions (Ellis [1])

(i) $(\epsilon + p) > 0$

(ii) $(\epsilon + 3p) > 0$.

From eqs (30) and (31) and the conditions (i), (ii) we get

$$4a^2 - \beta^2 > 0 \tag{32}$$

and

$$\left(\frac{a^2}{T^2} - \frac{\beta^2}{4T^2}\right)\frac{1}{\alpha^2 a^2} > 2\Lambda. \tag{33}$$

If $\Lambda > 0$, we have

$$\frac{4a^2 - \beta^2}{8a^2\,\alpha^2\,\Lambda} > T^2. \tag{34}$$

However if $\Lambda < 0$, condition (33) is obviously satisfied. We also notice that if Λ is negative $\epsilon > p$ in view of eqs (30) and (31). The scalar of expansion θ in the model (29) calculated for the flow vector v^i is given by

$$\theta = \frac{1}{aT}. \tag{35}$$

The rotation ω is identically zero and the non-vanishing components of the shear tensor σ_{ij} are given by

$$\sigma_{11} = -\frac{a}{3a^2T}, \tag{36}$$

$$\sigma_{22} = \frac{\cosh\phi}{6a} - \frac{\beta \sinh\phi}{4a\alpha} \tag{37}$$

$$\sigma_{23} = \left(\frac{4a\gamma + 3\beta^2}{12\, a\alpha}\right)\cosh\phi - \frac{(a+3\gamma)\,\beta \sinh\phi}{6a\alpha} \tag{38}$$

and

$$\sigma_{33} = \frac{a\beta^2 + 4a\gamma + 6\beta^2\gamma}{6a\alpha}\cosh\phi - \frac{\beta\,(16a\gamma + 12\gamma^2 + 3\beta^2)\sinh\phi}{6a\alpha}. \tag{39}$$

Since $T = 0$ is a singularity, we interpret this result that the model expands from a singular state at $T = 0$ and goes on expanding indefinitely. The red shift for the model (29) is given by

$$\frac{\lambda + \delta\lambda}{\lambda} = \frac{T_2^{1/2}\,(a^2 - a^2u^2)^{1/2}\left\{a\,\mathrm{sech}^{1/2}\phi + aU_ZT_1^{1/2}\right\}}{a^2T_1^{1/2}\,\mathrm{sech}^{1/2}\phi} \tag{40}$$

where u is the velocity of the source at the time of emission and U_Z is the Z-component of the velocity.

The flow vector v^i satisfies the equations of the geodesics $v^i_{;j}\,v^j = 0$ indicating that the line of flow are geodesics. The equations of the geodesics for the metric (29) are given by

$$\frac{d^2X}{dS^2} = 0, \tag{41}$$

$$\frac{d^2Y}{dS^2} + \frac{1}{2T}\left[1 + \frac{N\gamma}{a\beta}\right]\frac{dY}{dS}\frac{dT}{dS} + \frac{N}{4a\beta T}\frac{dZ}{dS}\cdot\frac{dT}{dS} = 0, \tag{42}$$

$$\frac{d^2Z}{dS^2} + \frac{N}{4aT}\left[(\beta\,\mathrm{sech}^2\phi - (2\gamma - \beta\tanh\phi)\left(3\tanh\phi - \frac{2\gamma}{\beta}\right)\right]\frac{dY}{dS}\frac{dT}{dS}$$

$$+\frac{1}{2T}\left[1-\frac{N\gamma}{a\beta}\right]\frac{dZ}{dS}\frac{dT}{dS}=0, \tag{43}$$

$$\frac{d^2T}{dS^2}+\frac{\beta a}{2a^2}(N\gamma+a\beta)\operatorname{sech}\phi+\frac{a}{8a^2}[(2\gamma-\beta\tanh\phi)^2\cdot$$

$$(2a\cosh\phi-N\sinh\phi)]\left(\frac{dY}{dS}\right)^2$$

$$+\frac{a^2\cosh\phi}{2a^2}\left[1-\frac{N}{2a}\tanh\phi\right]\left(\frac{dZ}{dS}\right)^2$$

$$+\frac{a}{2a^2}[N\beta\operatorname{sech}\phi+(2\gamma-\beta\tanh\phi)(2a\cosh\phi-N\sinh\phi)]$$

$$\frac{dY}{dS}\frac{dZ}{dS}=0. \tag{44}$$

If initially $dY/dS=0$ and $dZ/dS=0$ then dY/dS, dZ/dS and d^2T/dS^2 are zero for all time. Hence in this case

$$X = ms+n,$$

$$T = rs+q$$

$$Y = d$$

and

$$Z = h$$

where, d, h, m, n, r and q are arbitrary constants. The non-vanishing physical components of the conformal curvature tensor are given by

$$C_{(1212)}=C_{(1313)}=-\tfrac{1}{2}C_{(2323)}$$

$$=\frac{1}{12a^2a^2}\left(\frac{a^2}{T^2}-\frac{\beta^2}{4T^2}\right) \tag{45}$$

and

$$C_{(2434)}=-\frac{\beta^2\operatorname{sech}\phi\tanh\phi}{4a^2a^2T^2}. \tag{46}$$

Hence the flow vector is geodetic and the model represents an expanding, shearing but non-rotating universe, the gravitational field for which is of degenerate Petrov Type I. We also notice that if in (29) γ is put equal to zero the metric is simplified but the physical and geometrical features remain essentially the same.

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Preservation of stability and asymptotic behaviour of perturbed integrodifferential equations in a Banach space

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Abstract. In this paper stability and asymptotic behaviour of solutions of the integrodifferential system $x'(t) = A(t)\,x(t) + f\left(t, x(t), \int_{t_0}^{t} k(t, s, x(s))\,ds\right) + g(t, x(t))$ in a Banach space is related to that of the integrodifferential system $y'(t) = A(t)\,y(t) + f\left(t, y(t), \int_{t_0}^{t} k(t, s, y(s))\,ds\right)$ in a Banach space. The results obtained constitute a generalization of similar results for ordinary differential equations in a Banach space, which motivate the approach and proofs.

Keywords. Integrodifferential equations; Banach space; stability and asymptotic behaviour; asymptotic equivalence.

1. Introduction

Let B be a real Banach space. We are interested in the operators which are defined on subsets of B and take the values belonging to B. The operator A is linear if it is additive and homogeneous. Denote its domain of definition by $D[A]$. The operator A is said to be closed if the set of couples (x, Ax), $x \in D[A]$ is closed in the topological product $B \times B$. For a real α, suppose that $[I-\alpha A]^{-1}$ exists, where I is the identity operator and $D[(I-\alpha A)^{-1}]$ is dense in B. Let $R(\alpha, A) = [I-\alpha A]^{-1}$ and since A is closed we have

$$D[R(\alpha, A)] = B,$$

$$[I-\alpha A]\,R(\alpha, A)\,x = x, \quad (x \in B),$$

$$R(\alpha, A)\,[I-\alpha A]\,x = x, \quad (x \in D[A]),$$

where $R(\alpha, A)$ is called the resolvent of A. Let J denote the interval $t_0 \leqslant t < \infty$, $t_0 \geqslant 0$, and $|\cdot|$ denote any suitable norm in B. We denote by $C[E, N]$, the class of functions defined and continuous on E taking values in N, where E and N are any convenient spaces. Let R^+ denote the half line $0 \leqslant t < \infty$. In this paper we wish to study the behavioural relationships between the solutions of the integrodifferential system

$$(P) \quad x'(t) = A(t)\,x(t) + f\left(t, x(t), \int_{t_0}^{t} k(t, s, x(s))\,ds\right) + g(t, x(t)),\ x(t_0) = x_0,$$

for suitable perturbation terms g, and the solutions of the unperturbed integrodifferential system

$$(N) \quad y'(t)=A(t)\,y(t)+f\left(t, y(t), \int_{t_0}^{t} k(t, s, y(s))\,ds\right), \; y(t_0)=y_0,$$

where $x, y \in B$, $k \in C\,[J\times J\times B, B]$, $f \in C\,[J\times B\times B, B]$, $g \in C\,[J\times B, B]$, $f\left(t, 0, \int_{t_0}^{t} k(t, s, 0)\,ds\right)=0$, $g(t, 0)=0$, and the operators $A(t)$ satisfy the assumption (*) given below.

(*) Suppose that for each $t \in J$, $\{A(t)\}$ is one-parameter family of closed linear operators. Assume that for each $t \in J$, the resolvent set of $A(t)$ includes all positive real numbers, that the domain $R(\alpha, A(t))$ is dense in B, and that the domain $D[A(t)]$ of $A(t)$ is independent of t.

Let $x(t) \in D[A(t)]$ be a strongly continuous function defined for $t \in J$ and having a strong derivative $x'(t)$ in B. We shall call $x(t)$ a solution of the integrodifferential system (P) if it satisfies equation (P), with the initial value $x(t_0)=x_0$, $x_0 \in D[A(t_0)]$, $t_0 \geqslant 0$, for $t_0 < t < \infty$. Similarly we can define $y(t)$ as the solution of the integrodifferential system (N).

Many recent papers have dealt with the existence, uniqueness and other properties of solutions of some integral and integrodifferential equations in a Banach space, see for example, Dafermos [2], Friedman and Shinbrot [3], Lovelady [5], MacCamy and Wong [6], Marti [7], Miller [9], Pachpatte [11]-[14], to mention a few. In this paper, we shall assume, without further mention, that (P) and (N) have solutions with the initial values $x(t_0)=x_0$, $x_0 \in D[A(t_0)]$, $y(t_0)=y_0$, $y_0 \in D[A(t_0)]$ respectively for $t_0 \geqslant 0$, and limit our discussion to some problems in stability and asymptotic behaviour of solutions of (P) and (N). The integrodifferential systems considered in this paper contain, in particular, the abstract differential systems studied by Lakshmikantham and Leela [4, Ch. 12 Vol. II], and the abstract integrodifferential systems recently studied by this author in [11], [13], [14]. For other recent results on preserving stability and asymptotic behaviour of ordinary integral and integrodifferential equations see Bownds and Cushing [1], Miller [8] and Pachpatte [15].

2. Comparison theorems

In this section we state and prove some basic comparison theorems which can be used to study the behavioural relationships between the solutions of (P) and (N).

Theorem 1

Let the scalar functions $k_1 \in C\,[J\times J\times R^+, R]$, $W_1 \in C\,[J\times R^+\times R^+, R]$, $W_2 \in C\,[J\times R^+, R]$, and $W_2(t, r)$ is nondecreasing in r for each $t \in J$. Let $r(t)$ be any solution of

$$r'(t)=W_1\left(t, r(t), \int_{t_0}^{t} k_1(t, s, r(s))\,ds\right)+W_2(t, r(t)+\psi(t)), \; r(t_0)=r_0, \qquad (1)$$

existing to the right of t_0, where $\psi(t) \geqslant 0$ is a continuous function defined on J. Suppose for each $t \in J$

$$\lim_{h\to 0^+} R(h, A(t))\,(x-y)=x-y, \qquad (2)$$

for every x, y in the domain of the resolvent $R(h, A(t))$ which is assumed to be dense in B,

$$|R(h, A(t))\, x - R(h, A(t))\, y| \leqslant |x-y|\,(1-ah), \tag{3}$$

$$\left| f\left(t, x, \int_{t_0}^{t} k(t, s, x)\, ds\right) - f\left(t, y, \int_{t_0}^{t} k(t, s, y)\, ds\right)\right| \leqslant W_1\left(t, |x-y|, \int_{t_0}^{t} k_1(t, s, |x-y|)\, ds\right) \tag{4}$$

$$|g(t, x)| \leqslant W_2(t, |x|), \tag{5}$$

for every $x, y \in B$, and for all sufficiently small $h > 0$. Let $x(t)$ and $y(t)$ be solutions of (P) and (N) respectively for all $t \geqslant t_0$ such that $|x_0 - y_0| \leqslant r_0$, then

$$|x(t) - y(t)| \leqslant r(t),\ t \in J. \tag{6}$$

Proof. Define $m(t) = |x(t) - y(t)|$. For small $h > 0$ we have

$$\begin{aligned} m(t+h) \leqslant \Big| & x(t+h) - R(h, A(t))\, x(t) - h\left\{ f\left(t, x(t), \int_{t_0}^{t} k(t, s, x(s))\, ds\right) + g(t, x(t))\right\} - \left\{ y(t+h) - R(h, A(t))\, y(t)\right. \\ & \left. - hf\left(t, y(t), \int_{t_0}^{t} k(t, s, y(s))\, ds\right)\right\} \Big| + \Big| R(h, A(t))\, x(t) \\ & - R(h, A(t))\, y(t) + h\left\{ f\left(t, x(t), \int_{t_0}^{t} k(t, s, x(s))\, ds\right)\right. \\ & \left. - f\left(t, y(t), \int_{t_0}^{t} k(t, s, y(s))\, ds\right)\right\} + h\, g(t, x(t)) \Big|. \end{aligned} \tag{7}$$

Since for every $x \in D(A(t))$, $R(h, A(t))\,(I - hA(t))\, x = x$, it follows that

$$R(h, A(t))\, x = x + hA(t)\, x + h\,[R(h, A(t))\, A(t)\, x - A(t)\, x]. \tag{8}$$

Similarly, for every $y \in D(A(t))$, $R(h, A(t))\,(I - hA(t))\, y = y$, it follows that

$$R(h, A(t))\, y = y + h\, A(t)\, y + h\,[R(h, A(t))\, A(t)\, y - A(t)\, y]. \tag{9}$$

From (7), (8), (9) together with (P), (N), (2), (3), (4), (5) we obtain the inequality

$$\begin{aligned} \lim_{h \to 0^+} \sup \frac{1}{h}\,[m(t+h) - m(t)] \leqslant\ & W_1\left(t, m(t), \int_{t_0}^{t} k_1(t, s, m(s))\, ds\right) \\ & + W_2(t, m(t) + \psi(t)) \end{aligned} \tag{10}$$

where $\psi(t)=|y(t)|$. Suppose $r(t, \epsilon)$ be any solution of the integral equation

$$r'(t) = W_1\left(t, r(t), \int_{t_0}^{t} k_1(t, s, r(s)\, ds\right) + W_2(t, r(t) + \psi(t)) + \epsilon,$$

with the given initial condition $r(t_0, \epsilon)=r_0+\epsilon$, where $\epsilon>0$ is a sufficiently small number. Since $\lim_{\epsilon \to 0} r(t, \epsilon)=r(t)$, it is enough to show that

$$m(t) \leqslant r(t, \epsilon), \; t \geqslant t_0. \tag{11}$$

If the inequality (11) does not hold, there exists a first point $\beta \in J$ to the right of t_0 such that $m(\beta)=r(\beta, \epsilon)$ and that $m(t)>r(t, \epsilon)$ for $t>\beta$, then we have

$$\lim_{h \to 0^+} \sup \frac{1}{h}[m(\beta+h) - m(\beta)] \geqslant r'(\beta, \epsilon)$$

$$= W_1\left(\beta, r(\beta, \epsilon), \int_{t_0}^{\beta} k_1(\beta, s, r(s, \epsilon))\, ds\right) + W_2(\beta, r(\beta, \epsilon)+\psi(\beta))+\epsilon,$$

which contradicts (10). Hence the inequality (11) holds, and consequently the desired result follows.

Theorem 1 involves a comparison between solutions of the integrodifferential system (P) and (N) and the scalar integral equation (1). Assuming that (1) is stable in some sense, one can prove that the integrodifferential systems (P) and (N) have a corresponding relative stability property, see Lakshmikantham and Leela [4] and Pachpatte [13].

Our next theorem investigates the asymptotic equivalence between the solutions of (P) and (N) under some suitable conditions on the functions involved in (P) and (N).

Theorem 2

Let the functions k_1, W_1 and W_2 be as defined in theorem 1. Let $r(t)$ be any bounded solution of

$$r'(t) = W_1\left(t, r(t), \int_{t_0}^{t} k_1(t, s, r(s))\, ds\right) + W_2(t, r(t) + \psi(t)\, e^{\alpha t}),$$

$$r(t_0) = r_0, \tag{12}$$

existing to the right of t_0, where $\alpha>0$ is a constant and $\psi(t)\geqslant 0$ is a continuous function defined on J. Suppose for each $t \in J$

$$\lim_{h \to 0^+} R(h, A(t))(x - y) = x - y, \tag{13}$$

for every x, y in the domain of the resolvent $R(h, A(t))$ which is assumed to be dense in B,

$$|R(h, A(t))\, x - R(h, A(t))\, y| \leqslant |x - y|(1 - ah), \tag{14}$$

$$\left| f\left(t, x, \int_{t_0}^{t} k(t, s, x)\, ds\right) - f\left(t, y, \int_{t_0}^{t} k(t, s, y)\, ds\right) \right|$$

$$\leqslant \exp(-\alpha t)\, W_1\left(t, |x-y| \exp(\alpha t), \int_{t_0}^{t} k_1(t, s, |x-y| \exp(\alpha s))\, ds\right), \tag{15}$$

$$|g(t, x)| \leqslant \exp(-\alpha t)\, W_2(t, |x| \exp(\alpha t)), \tag{16}$$

for every $x, y \in B$, and for all sufficiently small $h>0$. Then

$$\lim_{t \to \infty} |x(t) - y(t)| = 0. \tag{17}$$

where $x(t)$ and $y(t)$ are the solutions of (P) and (N) respectively for all $t \geqslant t_0$ such that $|x_0 - y_0| \leqslant r_0$.

Proof. Defining $m(t) = \exp(\alpha t)\,|x(t)-y(t)|$, and following the similar argument as in the proof of theorem 1 we obtain

$$\exp(\alpha t)\,|x(t) - y(t)| \leqslant r(t),\ t \in J. \tag{18}$$

Since the solution $r(t)$ of (12) is bounded we can bring $\exp(\alpha t)$, $\alpha>0$, in (18) to the other side; then we obtain

$$\lim_{t \to \infty} |x(t)-y(t)| = 0.$$

The proof of the theorem is complete.

The above theorem establishes the asymptotic equivalence between the solutions of (P) and (N). In the special case where the integral terms in (P) and (N) are absent, then the theorem gives the asymptotic equivalence between two solutions of an ordinary differential system in a Banach space, see Lakshmikantham and Leela [4, Ch. 12, Vol. II].

In order to establish our next result we require the following hypothesis: (H) the solution $y(t, t_0, x_0)$ of (N) with the initial condition $y(t_0) = x_0$ exist for all $t \geqslant t_0$, unique, continuous with respect to the initial data and $|y(t, t_0, x_0)|$ is locally Lipschitzian in x_0.

For any $V \in C[J \times B, R^+]$ and any fixed $t \in (t_0, \infty)$ we define

$$D^+V(s, y(t, s, x)) = \lim_{h \to 0^+} \sup \frac{1}{h}\Big[V\Big(s+h, y(t, s+h, R(h, A(s))x + h\Big\{ f\Big(s, x, \int_{t_0}^{s} k(s, \tau, x)d\tau\Big) + g(s, x)\Big\}\Big) - V(s, y(t, s, x))\Big], \tag{19}$$

for $t_0 \leqslant s < t$ and $x \in B$.

Recently, Mitchell and Moore [10] have established a general comparison theorem which includes as a special case the well-known comparison theorem given in [4,

Vol. I, theorem 3.1.1]. In theorem 3 below we exploit the idea used by Mitchell and Moore [10] to establish a more general comparison theorem which relates the solutions of (P) and (N), and contains as a special case the well-known comparison theorem for ordinary differential equations in a Banach space given in [4, Vol. II, Ch. 12].

Theorem 3

Assume that

(i) for every x, y in the domain of the resolvent $R(h, A(t))$ which is assumed to be dense in B.

$$\lim_{h \to 0^+} R(h, A(t))x = x; \tag{20}$$

(ii) the hypothesis (H) holds;

(iii) $V \in C[J \times B, R^+]$, $V(s, x)$ is locally Lipschitzian in x and for $t_0 \leqslant s < t$, $x \in B$; and $k_1 \in C[J \times J \times R^+, R^+]$, $W_1 \in C[J \times R^+ \times R^+, R^+]$, $W_2 \in C[J \times R^+, R^+]$,

$$D^+V(s, y(t, s, x)) \leqslant W_1\left(s, V(s, y(t, s, x)), \int_0^s k_1(s, \tau, V(\tau, y(t, \tau, x)))d\tau\right)$$

$$+ W_2(s, V(s, y(t, s, x))) \tag{21}$$

(iv) the maximal solution $r(t, t_0, r_0)$ of

$$r'(t) = W_1\left(t, r(t), \int_{t_0}^t k_1(t, s, r(s)ds\right) + W_2(t, r(t)), \quad r(t_0) = r_0 \geqslant 0, \tag{22}$$

exists for $t \geqslant t_0$.

Then if $x(t) = x(t, t_0, x_0)$ is any solution of (P), we have

$$V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, r_0), \quad t \geqslant t_0, \tag{23}$$

provided $V(t_0, y(t, t_0, x_0)) \leqslant r_0$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (P) existing on $[t_0, \infty)$. Let $m(s) = V(s, y(t, s, x(s)))$ for $t_0 \leqslant s \leqslant t$, so that $m(t_0) = V(t_0, y(t, t_0 x_0)) \leqslant r_0$. Further, for small $h > 0$, we have

$$\begin{aligned} m(s+h) - m(s) &= V(s+h, y(t, s+h, x(s+h))) - V(s, y(t, s, x(s))) \\ &= V(s+h, y(t, s+h, x(s+h))) - V\left(s+h, y(t, s+h, R(h, A(s))x(s)\right. \\ &\quad + h\left\{f\left(s, x(s), \int_{t_0}^s k(s, \tau, x(\tau))\, d\tau\right) + g(s, x(s))\right\}\Big) \\ &\quad + V\left(s+h, y(t, s+h, R(h, A(s))x(s) + h\left\{f\left(s, x(s), \int_{t_0}^s k(s, \tau, x(\tau))d\tau\right)\right.\right. \\ &\quad \left.\left. + g(s, x(s))\right\}\right) - V(s, y(t, s, x(s))) \end{aligned}$$

$$\leqslant L\Big| y(t, s+h, x(s+h))-y(t, s+h, R(h, A(s))x(s)$$

$$+ h\Big\{f\Big(s, x(s), \int_{t_0}^{s} k(s, \tau, x(\tau))d\tau\Big)+g(s, x(s))\Big\}\Big|$$

$$+ V\Big(s+ h, y(t, s+h, R(h, A(s))x(s)+h\Big\{f\Big(s, x(s), \int_{t_0}^{s} k(s,\tau,x(\tau))\, d\tau\Big)$$

$$+ g(s, x(s))\Big\}\Big)-V(s, y(t, s,x(s)))$$

$$\leqslant LM\Big| x(s+h)-R(h, A(s))x(s)$$

$$+ h\Big\{f\Big(s, x(s) \int_{t_0}^{s} k(s, \tau, x(\tau))d\tau\Big)+g(s, x(s))\Big\}\Big|$$

$$+ V\Big(s+h, y(t, s+h, R(h, A(s))x(s)+h\Big\{f\Big(s, x(s), \int_{t_0}^{s} k(s,\tau, x(\tau))d\tau\Big)$$

$$+g(s, x(s))\Big\}\Big)-V(s, y(t, s, x(s))), \tag{24}$$

where L and M are positive constants. Since for every $x \in D[A(t)]$,

$R(h, A(t))\,(I-hA(t))\,x=x$, it follows that

$$R(h, A(t))x=x+hA(t)x+h[R(h, A(t))A(t)x-A(t)x]. \tag{25}$$

From (24), and (25) together with (P), (20) and (21), we obtain the inequality

$$D^{+}m(s)\leqslant W_1\Big(s, m(s), \int_{t_0}^{s} k_1(s, \tau, m(\tau))d\tau\Big)+W_2(s, m(s)). \tag{26}$$

Now, by following the similar argument as in the proof of theorem 1, we obtain the estimate

$$m(s)\leqslant r(s, t_0,r_0),\ t_0 \leqslant s<t, \tag{27}$$

provided $m(t_0)\leqslant r_0$. Since $m(t)=V(t, y(t, t,x(t)))=V(t, x(t, t_0, x_0))$, the desired result (23) follows from (27) by setting $s=t$. The proof is complete.

It is interesting to make the following observations:

(i) Taking $r_0=V(t_0, y(t, t_0, x_0))$, the inequality (23) becomes

$$V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, V(t_0, y(t, t_0, x_0))),\ t \geqslant t_0, \tag{28}$$

which shows the connection between the solutions of systems (P) and (N) in terms of the maximal solution of (22).

(ii) The trivial function $A(t)y+f\Big(t, y, \int_{t_0}^{t} k(t, s, y)\, ds\Big) \equiv 0$ is admissible in theorem 3 to yield the estimate (23) provided $V(t_0, x_0) \leqslant r_0$. In this case, $y(t, t_0, x_0)$

$= x_0$ and thus the hypothesis (H) is trivially verified. Since $y(t, s, x) = x$, the definition (19) reduces to

$$D^+ V(s, x) = \lim_{h \to 0^+} \sup \frac{1}{h}\Big[V\Big(s+h, R(h, A(s))x + h \Big\{ f\Big(s, x, \int_{t_0}^{s} k(s, \tau, x)\, d\tau\Big) + g(s, x)\Big\}\Big) - V(s, x)\Big], \tag{29}$$

which is the usual definition of the generalized derivative of the Lyapunov function relative to the system (P). Consequently, theorem 3 reduces, in this special case, when the integral term in (P) is absent, to a well-known comparison theorem given in Lakshmikantham and Leela [4, Ch. 12]. Thus it is clear that theorem 3 is a natural extension of known comparison results.

3. Stability and asymptotic behaviour

As an application of the comparison theorem 3, we shall consider in this section some results on stability and asymptotic behaviour of solutions of (P) by means of comparison with the behaviour of solutions of (N). The stability definitions (see definitions 3.2.1 and 3.2.8 in [4, Vol. I]) and the boundedness definitions (see definitions 3.13.1 in [4, Vol. I]) used in our subsequent discussion have analogous restatements for (P) and (N), see also Pachpatte [13]-[15].

Theorem 4

Assume that

(i) the conditions (i), (ii) and (iii) of theorem 3 hold;

(ii) $k_1 \in C[J \times J \times R^+, R^+]$, $W_1 \in C[J \times R^+ \times R^+, R^+]$, $W_2 \in C[J \times R^+, R^+]$;

$$W_1\Big(t, 0, \int_{t_0}^{t} k_1(t, s, 0)\, ds\Big) \equiv 0,\ W_2(t, 0) \equiv 0,$$

$$f\Big(t, 0, \int_{t_0}^{t} k(t, s, 0)\, ds\Big) \equiv 0,\ g(t, 0) \equiv 0;$$

(iii) the trivial solution of (N) is uniformly stable and the trivial solution $r=0$ of (22) is uniformly asymptotically stable;

(iv) $b(|x|) \leqslant V(t, x) \leqslant a(|x|)$, $(t, x) \in J \times B$,

where $a, b \in C[R^+, R^+]$, $a(u)$, $b(u)$ are increasing and $a(0)=0$.

Then the trivial solution of (P) is uniformly asymptotically stable.

Proof. Let $\epsilon > 0$, $t_0 \in J$ be given. The uniform stability of the trivial solution $r=0$ of (22) implies that given $b(\epsilon) > 0$, $t_0 \in J$ there exists a $\delta_1 = \delta_1(\epsilon)$ such that

$$r(t, t_0, r_0) < b(\epsilon),\ t \geqslant t_0 \text{ if } r_0 \leqslant \delta_1. \tag{30}$$

Let $\delta_2 = a^{-1}(\delta_1)$. Since $y=0$ of (N) is uniformly stable, given $\delta_2 > 0$, $t_0 \in J$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$|y(t, t_0, x_0)| < \delta_2,\ t \geqslant t_0, \text{ if } |x_0| < \delta. \tag{31}$$

We claim that $|x_0|<\delta$ also implies that $|x(t, t_0, x_0)|<\epsilon$, $t\geqslant t_0$, where $x(t, t_0, x_0)$ is any solution of (P). If this is not true, there would exist a solution $x(t, t_0, x_0)$ of (P) with $|x_0|<\delta$, a $t_1>t_0$ such that $|x(t_1, t_0, x_0)|\leqslant\epsilon$, $t_0\leqslant t\leqslant t_1$. Then, by theorem 3 we have

$$V(t, x(t, t_0, x_0))\leqslant r(t, t_0, V(t_0, y(t, t_0, x_0))), t_0\leqslant t\leqslant t_1. \tag{32}$$

Using (iv), (32), (31), and (30) we have

$$\begin{aligned} b(\epsilon)\leqslant V(t_1, x(t_1, t_0, x_0)) &\leqslant r(t_1, t_0, a(|y(t_1, t_0, x_0)|)) \\ &\leqslant r(t_1, t_0, a(\delta_2))\leqslant r(t_1, t_0, \delta_1)<b(\epsilon). \end{aligned}$$

This contradiction proves that $x=0$ of (P) is uniformly stable.

To show uniform asymptotic stability, we set $\epsilon=\epsilon_0$ and $\delta(\epsilon_0)=\delta_0$. Then from the foregoing argument, we have

$$b(|x(t, t_0, x_0)|)\leqslant V(t, x(t, t_0, x_0))\leqslant r(t, t_0, V(t_0, y(t, t_0, x_0)))$$

for all $t\geqslant t_0$ if $|x_0|<\delta_0$. From this it follows that

$$b(|x(t, t_0, x_0)|)\leqslant r(t, t_0, \delta_1(\epsilon_0)), t\geqslant t_0,$$

whenever $|x_0|<\delta_0$. Since $r(t, t_0, \delta_1)$ is a solution of (22) and b is nondecreasing, then the uniform asymptotic stability of $x=0$ follows from that of $r=0$, and the proof of the theorem is complete.

Theorem 5

Assume that

(i) the conditions (i), (ii), and (iii) of theorem 3 hold;

(ii) $b(|x|)\leqslant V(t, x)\leqslant a(t, |x|)$, $(t, x)\in J\times B$, where $a\in C[J\times R^+, R^+]$, $a(t, u)$ is increasing in u, $b\in C[R^+, R^+]$, $b(u)$ is increasing in u, $b(0)=0$ and $b(u)\to\infty$ as $u\to\infty$;

(iii) the system (N) is equibounded for $0\leqslant\alpha<\eta(t_0)$ and all solutions $r(t, t_0, r_0)$ of (22) satisfy $\lim_{t\to\infty} r(t, t_0, r_0)=0$.

Then every solution $x(t, t_0, x_0)$ of (P) satisfies $\lim_{t\to\infty} x(t, t_0, x_0)=0$.

Proof. Since the system (N) is equibounded, given $0<\alpha<\eta(t_0)$, $t_0\in J$, there exists a $\beta=\beta(t_0, \alpha)$ such that

$$|y(t, t_0, x_0)|<\beta, t\geqslant t_0 \text{ if } |x_0|\leqslant\alpha.$$

Let $|x_0|\leqslant\alpha$ so that $a(t_0, |y(t, t_0, x_0)|)\leqslant a(t_0, \beta)=\gamma(t_0, \alpha)$. This in view of theorem 3 and (ii), implies that

$$\begin{aligned} b(|x(t, t_0, x_0)|)\leqslant V(t, x(t, t_0, x_0))&\leqslant r(t, t_0, V(t_0, y(t, t_0, x_0))) \\ &\leqslant r(t, t_0, \gamma(t_0, \alpha)), t\geqslant t_0. \end{aligned}$$

Now by (iii) $\lim_{t\to\infty} r(t, t_0, \gamma(t_0, \alpha))=0$ so $\lim_{t\to\infty} b(|x(t, t_0, x_0)|)=0$ and it follows that $\lim_{t\to\infty} x(t, t_0, x_0)=0$, and the theorem is proved.

We note that the important special case which is covered by above theorems is the abstract differential system considered by Lakshmikantham and Leela [4, Ch. 12, Vol. II]. Consequently our results in this section may be regarded as generalizations to integrodifferential equations in a Banach space of the corresponding well-known results in [4].

4. Further results

Recently, Pachpatte [15] has studied the behavioural relationships between the solutions of two ordinary integrodifferential equations by using the integral inequalities recently established by this author. In this section we point out that the same technique can be applied equally well to study the behavioural relationships between the solutions of (P) and (N). In our subsequent discussion we shall follow without further mention the definitions and preliminaries used in [15].

Theorem 6

Assume that

(i) for each $t \in J$

$$\lim_{h \to 0^+} R(h, A(t))\,(x-y) = x-y$$

for every x, y in the domain of the resolvent $R(h, A(t))$ which is assumed to be dense in B,

$$| R(h, A(t))\, x - R(h, A(t))\, y | \leqslant | x-y | (1-ah)$$

for all sufficiently small $h>0$;

$$\text{(ii)} \quad \left| f\left(t, x, \int_{t_0}^{t} k(t, s, x)\, ds\right) - f\left(t, y, \int_{t_0}^{t} k(t, s, y)\, ds\right) \right| \leqslant L \left[| x-y | + \int_{t_0}^{t} h(s) | x-y | \, ds \right],$$

for every $x, y \in B$, and $|x|, |y| \leqslant a$, where a is a positive real number, L is a nonnegative constant and $h(t)$ is a nonnegative real—valued integrable function such that

$$\sup_{t \geqslant 0} \int_{t}^{t+\tau} [L + h(s)]\, ds \leqslant M(\tau),$$

where $M(\tau)$ is a continuous function for all $\tau > 0$;

(iii) there exists $r>0$ such that if $|x| \leqslant r$, then

$$| g(t, x) | \leqslant \gamma(t),$$

for every $x \in B$ and for all $t \geqslant 0$, where $\gamma(t)$ is a continuous function and

$$G(t) = \int_{t}^{t+1} \gamma(s)\, ds \to 0, \text{ as } t \to \infty;$$

(iv) the trivial solution of (N) is uniformly asymptotically stable.

Then there exist $T_0 \geqslant 0$ and $\delta_0 > 0$ such that if $t_0 \geqslant T_0$ and $|x_0| < \delta_0$, the solution $x(t, t_0, x_0)$ of (P) satisfies $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$. In particular, if $g(t, 0) = 0$, then the null solution of (P) is eventually uniformly asymptotically stable.

Proof. The proof is based on similar argument developed by the present author [15] and hence we indicate the main points. Constants associated with (N) regarding the stability of the null solution shall be starred and those associated with (P) shall not. Define

$$Q(t) = \sup \{G(s): t-1 \leqslant s < \infty\} \text{ on } [1, \infty).$$

Then $Q(t) \to 0$ as $t \to \infty$. By lemma 3.4 given by Strauss and Yorke [16].

$$\int_{t_0}^{t} \gamma(s)\, ds \leqslant \int_{t_0-1}^{t} G(s)\, ds \leqslant Q(t_0)\,(t-t_0+1),$$

if $t \geqslant t_0 \geqslant 1$. Without loss of generality we may assume $r \leqslant a \leqslant \delta_0^*$. Let $t_0 \geqslant 1$ and $|x_0| \leqslant r$. Then if $|x(t, t_0, x_0)|$, $|y(t, t_0, x_0)| \leqslant r$ on $[t_0, t_0+\tau]$ for some $\tau > 0$, for $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$, and defining $m(t) = |x(t)-y(t)|$, and following the similar argument as in the proof of theorem 1 we obtain the inequality

$$m'(t) \leqslant L\left[m(t) + \int_{t_0}^{t} h(n)\, m(n)\, dn\right] + \gamma(t). \tag{33}$$

Integrating both sides of (33) from t_0 to t, and using $m(t_0) = 0$, we have

$$\begin{aligned}
|x(t)-y(t)| &\leqslant \int_{t_0}^{t} L\left[|x(s)-y(s)| + \int_{t_0}^{s} h(n)\,|x(n)-y(n)|\, dn\right] ds \\
&\quad + \int_{t_0}^{t} \gamma(s)\, ds \\
&\leqslant \int_{t_0}^{t} L\left[|x(s)-y(s)| + \int_{t_0}^{s} h(n)\,|x(n)-y(n)|\, dn\right] ds \\
&\quad + Q(t_0)\,(t-t_0+1) \\
&= Q(t_0) + \int_{t_0}^{t} [L\,|x(s)-y(s)| + Q(t_0)]\, ds \\
&\quad + \int_{t_0}^{t} L\left(\int_{t_0}^{s} h(n)\,|x(n)-y(n)|\, dn\right) ds.
\end{aligned}$$

Using lemma 2 given in [15], we have

$$\begin{aligned}
|x(t)-y(t)| &\leqslant Q(t_0) + \int_{t_0}^{t} \left[Q(t_0) + L\left\{Q(t_0) \exp\left(\int_{t_0}^{s} [L+h(n)]\, dn\right)\right.\right. \\
&\quad \left.\left. + \int_{t_0}^{s} Q(t_0) \exp\left(\int_{n}^{s} [L+h(\rho)]\, d\rho\right) dn \right\}\right] ds.
\end{aligned}$$

The rest of the proof follows by the similar argument as in the proof of theorem 1 given by Pachpatte [15] and hence we omit the details.

In concluding this section we establish the following theorem on the eventual uniform asymptotic stability of the trivial solution of (P).

Theorem 7

Assume that all the conditions of theorem 6 are satisfied except that condition (ii) is replaced by

$$\text{(ii*)} \qquad \left| f\left(t, x, \int_{t_0}^{t} k(t, s, x)\, ds\right) - f\left(t, y, \int_{t_0}^{t} k(t, s, y)\, ds\right) \right| \leqslant \lambda(t) \left[|x-y| + \int_{t_0}^{t} \lambda(s)\, W(|x-y|)\, ds \right.$$

for every $x, y \in B$, and $|x|, |y| \leqslant r$, where r is a positive real number; $W(0)=0$; $u + W(u)$ is monotonically increasing in u; and $\lambda(t)$ is a nonnegative integrable function such that $\int_{t_0}^{\infty} \lambda(s)\, ds$ is finite.

Then there exist $T_0 \geqslant 0$ and $\delta_0 > 0$ such that if $t_0 \geqslant T_0$ and $|x_0| < \delta_0$, the solution $x(t, t_0, x_0)$ of (P) satisfies $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$ provided that

$$\int_0^1 \frac{ds}{s + W(s)}$$

is divergent. If $g(t, 0)=0$, then the null solution of (P) is eventually uniformly asymptotically stable.

The proof of this theorem follows by the similar argument as in the proof of theorem 2 given in Pachpatte [15] with suitable modifications and we leave the details to the reader.

It is interesting to note that the results established in this section are the further generalizations to integrodifferential equations in a Banach space of the recent results given in Pachpatte [15, theorems 1 and 2] which in turn are the further generalizations of the classical result given by Strauss and Yorke in [16, Theorem 5.1].

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On some fundamental integrodifferential inequalities and their discrete analogues

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Abstract. Some fundamental integrodifferential inequalities and their discrete analogues which can be used in the analysis of a class of integrodifferential and summary difference equations as handy tools are discussed.

Keywords. Integrodifferential inequalities; discrete analogues; summary difference equations

1. Introduction

Integral inequalities of considerable interest are associated with the names of Gronwall [3], Bellman [1], Bihari [2], and Willett and Wong [10]. The integral and discrete inequalities recently established by this author in [4]–[9] have been applied with considerable success to the study of many problems in the theory of differential and finite difference equations of the more general type. In this paper we wish to establish some new integrodifferential inequalities and their discrete analogues which can be used in applications as handy tools. It appears that the results obtained here are new to the literature, although their proofs use standard arguments.

2. Integrodifferential inequalities

In this section we wish to establish some interesting and useful integrodifferential inequalities which can be used in the study of many problems concerning the behaviour of solutions of a class of integrodifferential equations.

Theorem 1

Let $x(t)$, $x'(t)$, $a(t)$, and $b(t)$ be real-valued nonnegative continuous functions defined on $I=[0, \infty)$, for which the inequality

$$x'(t) \leqslant x(0)+\int_0^t a(s)(x(s)+x'(s))ds+\int_0^t a(s)\left(\int_0^s b(\tau)x'(\tau)d\tau\right)ds \tag{1}$$

holds for all $t \in I$. Then

$$x'(t) \leqslant x(0)\left[1+\int_0^t 2a(s)\exp\left(\int_0^s [1+a(\tau)+b(\tau)]\,d\tau\right)ds\right], \tag{2}$$

for all $t \in I$.

Proof. Define

$$m(t)=x(0)+\int_0^t a(s)(x(s)+x'(s))ds+\int_0^t a(s)\left(\int_0^s b(\tau)x'(\tau)d\tau\right)ds,$$

$$m(0)=x(0), \tag{3}$$

then (1) can be restated as

$$x'(t) \leqslant m(t). \tag{4}$$

Differentiating (3) and using (4) we have

$$m'(t)\leqslant a(t)\left[x(t)+m(t)+\int_0^t b(\tau)m(\tau)d\tau\right]. \tag{5}$$

Integrating both sides of (4) from 0 to t we have

$$x(t)\leqslant x(0)+\int_0^t m(s)ds. \tag{6}$$

Now, using (6) in (5) we obtain

$$m'(t)\leqslant a(t)\left[x(0)+m(t)+\int_0^t m(s)ds+\int_0^t b(\tau)m(\tau)d\tau\right]. \tag{7}$$

If we put

$$u(t)=x(0)+m(t)+\int_0^t m(s)ds+\int_0^t b(\tau)m(\tau)d\tau,\ u(0)=2x(0) \tag{8}$$

then it follows by differentiating (8) and using the facts that $m'(t) \leqslant a(t)u(t)$ and $m(t) \leqslant u(t)$ from (7) and (8) we see that the inequality

$$u'(t) \leqslant [1+a(t)+b(t)]u(t)$$

is satisfied, which implies the estimation for $u(t)$ such that

$$u(t)\leqslant 2x(0)\exp\left(\int_0^t [1+a(s)+b(s)]ds\right),$$

since $u(0)=2x(0)$. Substituting this value of $u(t)$ in (7) we have

$$m'(t)\leqslant 2x(0)a(t)\exp\left(\int_0^t [1+a(s)+b(s)]ds\right),$$

for all $t \in I$, which implies the estimate for $m(t)$ such that

$$m(t)\leqslant x(0)\left[1+\int_0^t 2a(s)\exp\left(\int_0^t [1+a(\tau)+b(\tau)]d\tau\right)ds\right].$$

Now, substituting this value of $m(t)$ in (4) we obtain the desired bound in (2).

It is interesting to note that the advantage of inequality (2) over inequality (1) becomes apparent if we consider $x(0)$, $a(t)$, and $b(t)$ as known functions and $x(t)$ and $x'(t)$ as unknown, that is, inequality (2) gives us a completely known function which majorizes $x'(t)$ and consequently $x(t)$ after integration.

Another interesting and useful integrodifferential inequality is embodied in the following theorem.

Theorem 2

Let $x(t)$, $x'(t)$, $a(t)$ and $b(t)$ be real-valued nonnegative continuous functions defined on I. for which the inequality

$$x'(t) \leqslant x(0) + \int_0^t a(s)(x(s)+x'(s))ds + \int_0^t a(s)\left(\int_0^s b(\tau)x'^{\alpha}(\tau)d\tau\right)ds, \tag{9}$$

holds for all $t \in I$, where $0 \leqslant \alpha < 1$. Then

$$x'(t) \leqslant x(0) + \int_0^t a(s) \exp\left(\int_0^s [1+a(\tau)]\, d\tau\right)\Big[(2x(0))^{1-\alpha} + (1-\alpha)\int_0^s b(\tau) \exp\left(-(1-\alpha)\int_0^\tau [1+\alpha(n)]\, dn\right) d\tau\Big]^{1/1-\alpha} ds, \tag{10}$$

for all $t \in I$.

The proof of this theorem follows by the similar argument as in the proof of theorem 1. We omit the details.

We next establish a slightly different version of theorem 1 which can be used in some applications.

Theorem 3

Let $x(t)$, $x'(t)$, $a(t)$, and $b(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality

$$x'(t) \leqslant x(0) + \int_0^t a(s)\,(x(s)+x'(s))\, ds + \int_0^t a(s)\left(\int_0^s b(\tau)\,(x(\tau) + x'(\tau))\, d\tau\right) ds \tag{11}$$

holds for all $t \in I$. Then

$$x'(t) \leqslant x(0)\Big[1 + \int_0^t a(s)\Big\{2 \exp\left(\int_0^s [1+a(\tau)]\, d\tau\right) + \int_0^s 3b(\tau) \exp\left(\int_\tau^s [1+a(n)]\, dn\right) \exp\left(\int_0^\tau [2+a(n)+b(n)]\, dn\right) d\tau\Big\}\, ds\Big] \tag{12}$$

for all $t \in I$.

The details of the proof of this theorem follows by an argument similar to that in the proof of theorem 1 with suitable modifications, and we leave the details to the reader.

We note that the inequalities established in theorems 1, 2, and 3 give us the estimates on $x'(t)$ and consequently on $x(t)$ after integration, which are different from the estimates which are already available in the literature.

Before leaving this section, we indicate a simple application of our theorem 3 to obtain the bound for the solutions of some integrodifferential equations. There are many possible applications of the inequalities established in this section, but the one presented here is sufficient to convey the importance of our results.

Consider the Volterra integrodifferential equation of the form

$$x'(t) = f(t) + \int_0^t H\left[t, s, x(s), x'(s), \int_0^s k(s, \tau, x(\tau), x'(t))\, d\tau\right] ds, \tag{13}$$

where x, x', f, k, and H are the elements of R^n, an n-dimensional Euclidean space, and continuous on the respective domains of their definitions. Let $|\cdot|$ denote some convenient norm on R^n. Assume that the functions f, k, and H in (13) satisfy

$$|f(t)| \leqslant x(0),\ t \in I, \tag{14}$$

$$|k(t, s, x(s), x'(s))| \leqslant b(s)\ [|x(s)| + |x'(s)|],\ 0 \leqslant s \leqslant t < \infty, \tag{15}$$

$$|H[t, s, x(s), x'(s), u]| \leqslant a(s)\ [|x(s)| + |x'(s)| + |u|],\ 0 \leqslant s \leqslant t < \infty, \tag{16}$$

where $a(t)$ and $b(t)$ are continuous functions defined on I and $x(o)$ is a nonnegative constant. Now, using (14), (15), and (16) in (13) and using theorem 3, we can establish very easily the bound on $x'(t)$ and consequently on the solution $x(t)$ of (13) in terms of the known functions.

3. Discrete analogues

In this section we wish to establish the discrete analogues of the integrodifferential inequalities established in section 2. The inequalities proved in this section can be used in the analysis of summary difference equations of the more general type. Before giving the main results in this section, we first introduce a few of the basic notions involved in our subsequent discussion. Let N be a set of points n_0+k ($k=0, 1, 2, \ldots$), where $n_0 \geqslant 0$ is a given integer. The expression $\sum_{s=n_0}^{n-1} b(s)$ represents a solution of the linear difference equation $\triangle x(n)=b(n)$ for all $n \in N$ under the initial condition $x(n_0)=0$, where $\triangle$ is the operator defined by $\triangle x(n)=x(n+1)-x(n)$. It is supposed that $\sum_{s=n_0}^{n_0-1} b(s)=0$. The expression $\prod_{s=n_0}^{n-1} c(s)$ represents a solution of the linear difference equation $x(n+1)=c(n)\, x(n)$ for all $n \in N$ under the initial condition $x(n_0)=1$. It is supposed that $\prod_{s=n_0}^{n_0-1} c(s)=1$.

We now state and prove the discrete analogue of the integrodifferential inequality established in theorem 1.

Theorem 4

Let $x(n)$, $\triangle x(n)$, $a(n)$, and $b(n)$ be real-valued nonnegative functions defined on N, for which the inequality

$$\triangle x(n) \leqslant x(n_0) + \sum_{s=n_0}^{n-1} a(s)\,(x(s)+\triangle x(s)) + \sum_{s=n_0}^{n-1} a(s)\Big(\sum_{t=n_0}^{s-1} b(t)\,\triangle x(t)\Big), \tag{17}$$

holds for all $n \in N$. Then

$$\triangle x(n) \leqslant x(n_0)\Big[1 + \sum_{s=n_0}^{n-1} 2a(s)\,\Pi_{t=n_0}^{s-1}\,[2+a(t)+b(t)]\Big], \tag{18}$$

for all $n \in N$.

Proof. Define

$$m(n) = x(n_0) + \sum_{s=n_0}^{n-1} a(s)\,(x(s)+\triangle x(s)) + \sum_{s=n_0}^{n-1} a(s)\Big(\sum_{t=n_0}^{s-1} b(t)\,\triangle x(t)\Big), \tag{19}$$

then (17) can be restated as

$$\triangle x(n) \leqslant m(n). \tag{20}$$

From (19) and (20) we have

$$\triangle m(n) \leqslant a(n)\Big[x(n)+m(n) + \sum_{t=n_0}^{n-1} b(t)\,m(t)\Big]. \tag{21}$$

Again from (20) we have

$$x(n) \leqslant x(n_0) + \sum_{t=n_0}^{n-1} m(t), \tag{22}$$

Now, using (22) in (21) we obtain

$$\triangle m\,(n) \leqslant a(n)\Big[x(n_0)+m(n) + \sum_{t=n_0}^{n-1} m(t) + \sum_{t=n_0}^{n-1} b\,(t)m(t)\Big]. \tag{23}$$

If we put

$$\begin{aligned} u(n) &= x(n_0)+m(n) + \sum_{t=n_0}^{n-1} m(t) + \sum_{t=n_0}^{n-1} b(t)m(t), \\ u(n_0) &= x(n_0)+m(n_0), \end{aligned} \tag{24}$$

then it follows from (24) and using the facts that $\triangle m(n) \leqslant a(n)u(n)$ and $m(n) \leqslant u(n)$ from (23) and (24) we see that the inequality

$$u(n+1) \leqslant [2+a(n)+b(n)]u(n)$$

is satisfied, which implies the estimation for $u(n)$ such that

$$u(n) \leqslant 2x(n_0)\,\Pi_{s=n_0}^{n-1}\,[2+a(s)+b(s)].$$

Since $u(n_0) = 2x(n_0)$. Substituting this value of $u(n)$ in (23) we have

$$\triangle m(n) \leqslant 2x(n_0)a(n) \prod_{s=n_0}^{n-1} [2+a(s)+b(s)],$$

for all $n \in N$, which implies the estimate for $m(n)$ such that

$$m(n) \leqslant x(n_0)\Big[1 + \sum_{s=n_0}^{n-1} 2a(s) \prod_{t=n_0}^{s-1} [2+a(t)+b(t)]\Big].$$

Now, substituting this value of $m(n)$ in (20) we obtain the desired bound in (18).

We next state the discrete analogues of theorems 2 and 3 established in section 2.

Theorem 5

Let $x(n)$, $\triangle x(n)$, $a(n)$, and $b(n)$ be real-valued nonnegative functions defined on N, for which the inequality

$$\triangle x(n) \leqslant x(n_0) + \sum_{s=n_0}^{n-1} a(s)(x(s)+\triangle x(s)) + \sum_{s=n_0}^{n-1} a(s)\Big(\sum_{t=n_0}^{s-1} b(t)[\triangle x(t)]^{\alpha}\Big), \tag{25}$$

holds for all $n \in N$, where $0 \leqslant \alpha < 1$, then

$$\triangle x(n) \leqslant x(n_0) + \sum_{s=n_0}^{n=1} a(s) \prod_{t=n_0}^{s-1} [2+a(t)] \Big[(2x(n_0))^{1-\alpha} + (1-\alpha)\sum_{t=n_0}^{s-1} b(t) \prod_{\tau=n_0}^{t} [2+a(\tau)]^{\alpha-1}\Big]^{1/1-\alpha}, \tag{26}$$

for all $n \in N$.

The proof of this theorem follows by the similar argument as in the proof of theorem 4 together with the proof of theorem 4 recently given by Pachpatte in [9]. We omit the details.

Theorem 6

Let $x(n)$, $\triangle x(n)$, $a(n)$, and $b(n)$ be real-valued nonnegative functions defined on N, for which the inequality

$$\triangle x(n) \leqslant x(n_0) + \sum_{s=n_0}^{n-1} a(s)(x(s)+\triangle x(s)) + \sum_{s=n_0}^{n-1} a(s)\Big(\sum_{t=n_0}^{s-1} b(t)(x(t)+\triangle x(t))\Big), \tag{27}$$

holds for all $n \in N$. Then

$$\triangle x(n) \leqslant x(n_0)\Big[1 + \sum_{s=n_0}^{n-1} 2a(s)\Big\{\prod_{t=n_0}^{s-1}[2+a(t)] + \sum_{t=n_0}^{s-1} b(t) \prod_{\tau=t+1}^{s-1}[2+a(\tau)] \prod_{\tau=n_0}^{t-1}[3+a(\tau)+b(\tau)]\Big\}\Big], \tag{28}$$

for all $n \in N$.

By setting $m(n)$ equal to the expression on the right side of (27) and following the similar argument as in the proof of theorem 4 with suitable modifications we obtain the desired bound in (28).

In concluding this paper we note that the discrete inequality established in theorem 4 can be used to establish a bound on the solution of a summary difference equation of the form

$$\Delta z(n) = f(n) + \sum_{s=n_0}^{n-1} H\Big[n, s, z(s), \Delta z(s), \sum_{t=n_0}^{s-1} k(s, \tau, \Delta z(\tau))\Big], \tag{29}$$

under some suitable conditions on the functions involved in (29). In our future papers we illustrate further applications of our inequalities to questions of asymptotic equivalence and stability problems in the theory of integrodifferential and summary difference equations.

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Heat transfer for laminar flow through parallel porous disks of different permeability

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Abstract. The problem of temperature distribution and heat transfer for laminar flow through two parallel porous disks of different permeability, has been investigated when the flow is entirely due to injection and/or suction at the two disks. Viscous dissipation terms have been included in the energy equation and the injection and/or suction velocities at the two disks are assumed to be small. The boundaries are kept at constant temperatures. The variation of temperature and Nusselt numbers at the two disks has been graphically depicted for various values of the injection and suction velocities.

Keywords. Porous; permeability; temperature; Nusselt number.

1. Introduction

Problems of flow and heat transfer between two parallel porous or non-porous disks are of great importance in the design of thrust bearings, radial diffusers, etc. Berman [1], Sellars [7], Yuan [11] and Yuan and Finkelstein [12] have studied the flow through porous boundaries with uniform suction or injection. Terill and Shrestha [9] have examined the laminar flow through parallel and uniformly porous walls of different permeability while the same problem for an annulus was investigated by Verma and Bansal [10]. Gaur [3] has studied viscous incompressible flow between two parallel porous rotating disks for small Reynolds number defined in terms of the angular velocities of the two disks. The problem of radial flow of a viscous incompressible fluid between two stationary uniformly porous disks has been investigated by Terrill and Cornish [8]. They have obtained solutions for small as well as large suction Reynolds number.

Inman [5] has discussed the effect of variation of the cross-flow velocity on the temperature distribution and heat transfer for flow in an annulus with porous walls, under the assumption that the rate of injection of the fluid at one boundary is equal to the rate of suction of the fluid at the other. Gaur [4] has studied the unsteady temperature distribution for laminar flow in a porous straight channel while Rajvanshi [6] has obtained heat transfer from two parallel coaxial disks rotating at different speeds, when there is a flow source at the axis of the rotation.

In the present investigation, we have obtained the solution for the energy equation, when the viscous dissipation terms have been included. It is assumed that there is injection at the lower disk and suction at the upper and that the flow is entirely due to

this injection and suction. Velocity components have been taken from reference [8] when the injection and suction velocities at the lower and upper disks are unequal but small. Regular perturbation scheme is used for expressing temperature. The disks are maintained at fixed temperatures and the effects of injection/suction velocities on the temperature and Nusselt numbers at the two disks are shown graphically.

2. Problem formulation

The fluid is contained between two infinite parallel porous disks and the flow takes place entirely due to the injection and suction velocities at the two disks. The surfaces of the two disks are situated in $z=-d$ and $z=+d$ respectively. We make use of the cylindrical polar co-ordinate system (r, θ, z). The axi-symmetric form of the energy equation, in this coordinate system, is

$$\rho C_p\left(u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right)=K\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\phi, \tag{1}$$

where u and w are the velocities in the r and z directions respectively. T, ρ, C_p and K are temperature, density, specific heat at constant pressure and coefficient of thermal conductivity of the fluid respectively. The viscous dissipation function ϕ is given by

$$\phi=2\mu\left[\left(\frac{\partial u}{\partial r}\right)^2+\frac{u^2}{r^2}+\left(\frac{\partial w}{\partial z}\right)^2\right]+\mu\left(\frac{\partial u}{\partial z}\right)^2. \tag{2}$$

The boundary conditions, are

$$\left.\begin{aligned} z&=-d,\ T=T_1\\ \text{and}\qquad\qquad\\ z&=+d,\ T=T_2\end{aligned}\right\} \tag{3}$$

where T_1 and T_2 are some constant temperatures.

Introducing the following non-dimensional quantities

$$\bar{r}=\frac{r}{d},\ \bar{z}=\frac{z}{d},\ \bar{u}=\frac{\rho u d}{\mu},\ \bar{w}=\frac{\rho w d}{\mu}$$

and

$$\bar{T}=\frac{T-T_1}{T_2-T_1}, \tag{4}$$

eq. (1) using (2) takes the form

$$\left(u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right)=\frac{1}{P_r}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)$$
$$+2E\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{u}{r}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+\tfrac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2\right], \tag{5}$$

where the bars from the non-dimensional quantities have been dropped; $p_r=(\mu C_p)/K$, is the Prandtl number and

$$E=\frac{\mu^2}{\rho^2 d^2 C_p (T_2-T_1)}, \text{ is the Eckert number.}$$

The boundary conditions, now become

$$\left.\begin{aligned} z=-1,\ T=0 \\ z=+1,\ T=1 \end{aligned}\right\} \quad \text{and} \tag{6}$$

It is assumed that there is injection at the lower disk and suction at the upper and that the non-dimensional velocities of injection and suction are respectively V_1 and V_2.

3. Method of solution

For small values of the injection and suction velocities V_1 and V_2, the velocity components w and u are given by

$$w=h(z)=\tfrac{1}{4}(V_1-V_2)(z^3-3z)+\tfrac{1}{2}(V_1+V_2)$$

$$+\frac{(V_1-V_2)^2}{2240}(z^7-21z^5+39z^3-19z)$$

$$+\frac{(V_1^2-V_2^2)}{32}(z^2-1)^2,$$

and

$$u=-\frac{r}{2}h'(z), \tag{7}$$

where V_1 and V_2 have been assumed to be small. The prime denotes differentiation with respect to z.

Substituting (7), in (5), we have

$$-\frac{r}{2}h'\frac{\partial T}{\partial r}+h\frac{\partial T}{\partial z}=\frac{1}{P_r}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)$$

$$+E(3h'^2+\tfrac{1}{4}r^2h''^2). \tag{8}$$

Equation (8) readily suggests that T, should be of the form

$$T=r^2T_2(z)+T_0(z). \tag{9}$$

Substituting (9) in (8) and equating the coefficients of r^2 and r^0 from the two sides of the equation, we get

$$-h'T_2 + hT_2' = \frac{1}{P_r} T_2'' + \frac{E}{4} h''^2,$$

and

$$hT_0' = \frac{1}{P_r}(4T_2 + T_0'') + 3Eh'^2. \tag{10}$$

The boundary conditions now become

$$\left.\begin{aligned} z = -1,\ T_2 = T_0 = 0 \\ z = 1,\ T_2 = 0,\ T_0 = 1 \end{aligned}\right\} \tag{11}$$

Let

$$T_2 = T_{2,0} + V_1 T_{2,1} + V_2 T_{2,2} + V_1^2 T_{2,11} + V_1 V_2 T_{2,12} + V_2^2 T_{2,22} + \ldots$$

$$T_0 = T_{0,0} + V_1 T_{0,1} + V_2 T_{0,2} + V_1^2 T_{0,11} + V_1 V_2 T_{0,12} + V_2^2 T_{0,22} + \ldots$$

and

$$h = V_1 h_1 + V_2 h_2 + V_1^2 h_{11} + V_1 V_2 h_{12} + V_2^2 h_{22} + \ldots . \tag{12}$$

Substituting (12) in (10), we obtain the following two sets of differential equations

I set

$$\frac{1}{P_r} T''_{2,0} = 0,$$

$$-h'_1 T_{2,0} + h_1 T'_{2,0} = \frac{1}{P_r} T''_{2,1},$$

$$-h'_2 T_{2,0} + h_2 T'_{2,0} = \frac{1}{P_r} T''_{2,2},$$

$$-h'_1 T_{2,1} - h'_{11} T_{2,0} + h_1 T'_{2,1} + h_{11} T'_{2,0} = \frac{1}{P_r} T''_{2,11} + \frac{E}{4} h_1''^2,$$

$$-h'_1 T_{2,2} - h'_2 T_{2,1} - h'_{12} T_{2,0} + h_1 T'_{2,2} + h_2 T'_{2,1} + h_{12} T'_{2,0}$$

$$= \frac{1}{P_r} T''_{2,12} + \frac{E}{2} h''_1 h''_2,$$

and

$$-h'_2 T_{2,2} - h'_{22} T_{2,0} + h_2 T'_{2,2} + h_{22} T'_{2,0} = \frac{1}{P_r} T''_{2,22} + \frac{E}{4} h''^2_2. \tag{13}$$

II set

$$4T_{2,0} + T''_{0,0} = 0,$$

$$h_1 T'_{0,0} = \frac{1}{P_r}(4T_{2,1} + T''_{0,1}),$$

$$h_2 T'_{0,0} = \frac{1}{P_r}(4T_{2,2} + T''_{0,2}),$$

$$h_1 T'_{0,1} + h_{11} T'_{0,0} = \frac{1}{P_r}(4T_{2,11} + T''_{0,11}) + 3Eh'^2_1$$

$$h_1 T'_{0,2} + h_2 T'_{0,1} + h_{12} T'_{0,0} = \frac{1}{P_r}(4T_{2,12} + T''_{0,12}) + 6Eh'_1 h'_2,$$

and

$$h_2 T'_{0,2} + h_{22} T'_{0,0} = \frac{1}{P_r}(4T_{2,22} + T''_{0,22}) + 3Eh'^2_2. \tag{14}$$

The boundary conditions take the form

$$\begin{aligned} z = -1: \quad & T_{2,0} = T_{2,1} = T_{2,2} = T_{2,11} = T_{2,12} = T_{2,22} = 0 \\ & T_{0,0} = T_{0,1} = T_{0,2} = T_{0,11} = T_{0,12} = T_{0,22} = 0 \\ z = +1: \quad & T_{2,0} = T_{2,1} = T_{2,2} = T_{2,11} = T_{2,12} = T_{2,22} = 0 \\ & T_{0,0} = 1,\ T_{0,1} = T_{0,2} = T_{0,11} = T_{0,12} = T_{0,22} = 0. \end{aligned} \tag{15}$$

As h_1, h_2, h_{11}, h_{12} and h_{22} are all known from (7), (13) and then (14) can be solved [making use of (15)] and the expressions for T_2 and T_0 obtained, which when substituted in (9), will give

$$\begin{aligned} T = r^2 & \frac{3EP_r}{64}(V_1 - V_2)^2 (1 - z^4) + \frac{1}{2}(1 + z) - \frac{V_1 P_r}{160}(1 - z^2)(20 - 9z + z^3) \\ & - \frac{V_2 P_r}{160}(1 - z^2)(20 + 9z - z^3) + V_1^2 (1 - z^2) \\ & \times \left[\frac{EP_r}{160}(113 - 37z^2 + 8z^4) - \frac{P_r}{4480 \times 360}(9240 - 613z - 3360z^2 + 527z^3 \right. \\ & + 840z^4 - 175z^5 + 5z^7) + \frac{P_r^2}{640 \times 2520}(10920 - 32209z + 33600z^2 \\ & \left. - 9949z^3 - 4200z^4 + 2525z^5 - 175z^7)\right] - V_1 V_2 (1 - z^2) \end{aligned}$$

$$\times\left[\frac{EP_r}{80}(113-37z^2+8z^4)+\frac{P_r}{2240\times360}(613z-527z^3+175z^5-5z^7)\right.$$

$$\left.+\frac{P_r^2}{320\times2520}(34991z-9949z^3+2525z^5-175z^7)\right]+V_2^2(1-z^2)$$

$$\times\left[\frac{EP_r}{160}(113-37z^2+8z^4)+\frac{P_r}{4480\times360}(9240+613z-3360z^2\right.$$

$$-527z^3+840z^4+175z^5-5z^7)-\frac{P_r^2}{640\times2520}(10920+32209z$$

$$\left.+33600z^2+9949z^3-4200z^4-2525z^5+175z^7)\right], \tag{16}$$

which is correct up to the second order terms in V_1 and V_2.

For $V_2=-V_1$, that is when there is uniform injection of equal magnitude at the two disks, (16) is in agreement with the result of Chaudhary and Gaur [2].

The rate of heat transfer is expressed in terms of Nusselt number, which in terms of the dimensional quantities, is given by

$$(Nu)=\frac{2dQ^*}{K(T_2-T_1)}, \tag{17}$$

where $$Q^*=\frac{1}{\pi(r^2-r_0^2)}\int_{r_0}^{r}(2\pi rq)\,dr,$$

and $$q=-K\frac{\partial T}{\partial z},$$

where the quantities Q^* and q are self evident from their expressions, r_0 is the distance of a given point on the either disk from the centre.

Making use of the non-dimensional quantities, as given in (4) and dropping bars, obtaining the value of Q^* for $z=-1$ and $z=+1$ [making use of (16)], Nusselt number at the lower disk $[(Nu)_{-1}]$ and Nusselt number at the upper disk $[(Nu)_{+1}]$ are given by

$$(Nu)_{-1}=-\frac{3}{16}EP_r(V_1-V_2)^2r_0^2(\lambda^2+1)-2\left\{\frac{1}{2}-\frac{7}{20}V_1P_r-\frac{3}{20}V_2P_r\right.$$

$$+V_1^2\left(\frac{21}{20}EP_r-\frac{109}{35\times360}P_r+\frac{313}{3150}P_r^2\right)$$

$$-V_1V_2\left(\frac{21}{10}EP_r-\frac{1}{35\times45}P_r-\frac{107}{1575}P_r^2\right)$$

$$\left.+V_2^2\left(\frac{21}{20}EP_r+\frac{101}{35\times360}P_r-\frac{1}{1575}P_r^2\right)\right\} \tag{18}$$

and

$$(Nu)_{+1} = \frac{3}{16} EP_r (V_1 - V_2)^2 r_0^2 (\lambda^2 + 1) - 2 \left\{ \frac{1}{2} + \frac{3}{20} V_1 P_r + \frac{7}{20} V_2 P_r \right.$$
$$- V_1^2 \left(\frac{21}{20} EP_r - \frac{101}{35 \times 360} P_r + \frac{1}{1575} P_r^2 \right)$$
$$+ V_1 V_2 \left(\frac{21}{10} EP_r + \frac{1}{35 \times 45} P_r + \frac{107}{1575} P_r^2 \right)$$
$$\left. - V_2^2 \left(\frac{21}{20} EP_r + \frac{109}{35 \times 360} P_r - \frac{313}{3150} P_r^2 \right) \right\},$$

where $\lambda = r/r_0$. (19)

For $V_2 = -V_1$, (18) and (19) are also in agreement to the corresponding results in [2].

4. Numerical discussion

Figure 1 shows temperature profiles for various values of V_1 and V_2. When $V_1 = 0{\cdot}4$ and $V_2 = 0{\cdot}2$, that is injection at the lower disk and suction at the upper, the temperature rises from zero at the lower disk to one at the upper disk, remaining more than one throughout the region of the fluid except in the neighbourhood of the lower disk. $V_1 = -0{\cdot}4$ and $V_2 = 0{\cdot}2$, that is when there is suction at both the disks, at every point

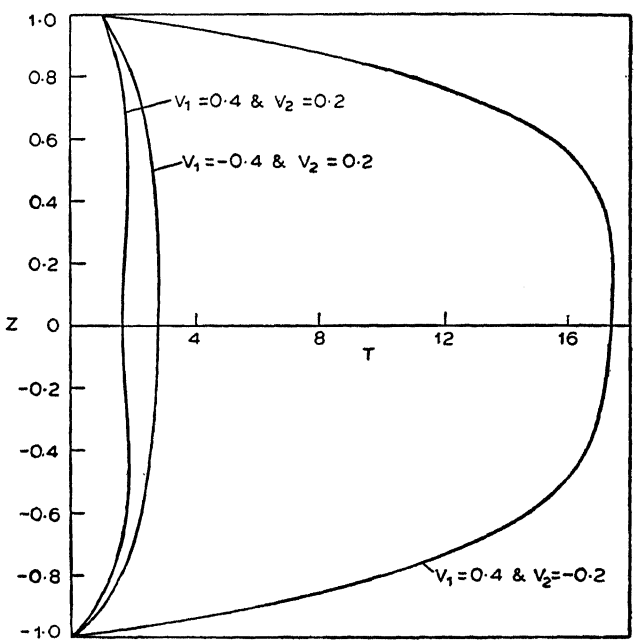

Figure 1. Temperature distribution for various values of injection/suction velocities.

of the fluid, the temperature is more than its corresponding part, when V_1=0·4 and V_2=0·2. The temperature in this case is maximum in the central plane. Injection at both the disks ($V_1 = 0{\cdot}4$, $V_2 = -0{\cdot}2$) gives rise to a sharp increase in the temperature, which increases as the fluid particle moves from the lower disk to the central plane and then decreases from central plane to the upper disk. The temperature profile in this case is almost parabolic.

Figure 2 depicts the variation of Nusselt number at the lower disk against $\lambda(=r/r_0)$ Injection at the lower and suction at the upper disk (V_1=0·4, V_2=0·2) results in the decrease in the Nusselt number, but this decrease is much slower as we advance from

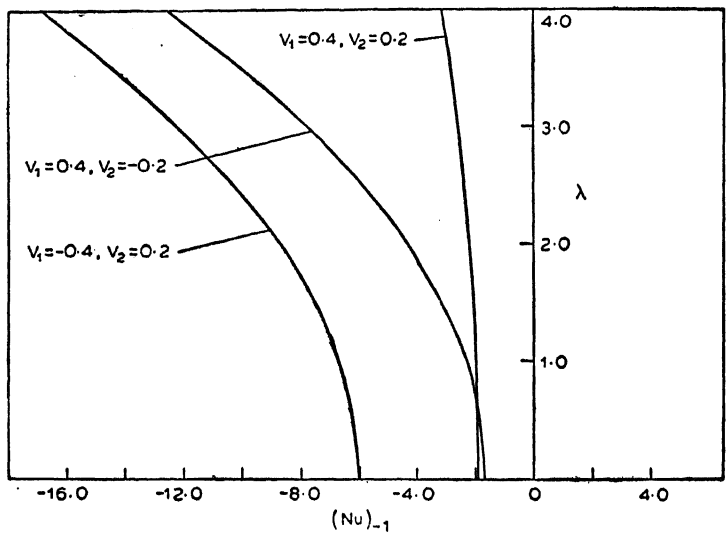

Figure 2. Nusselt number at the lower disk for various values of injection/suction velocities.

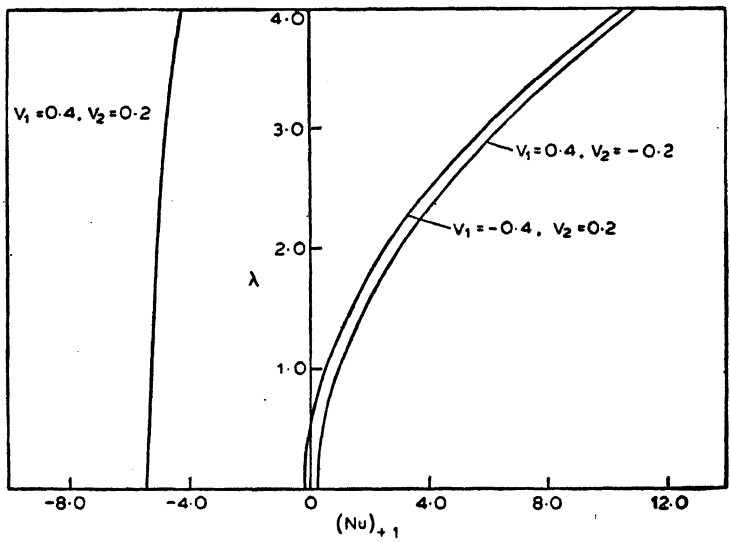

Figure 3. Nusselt number at the upper disk for various values of injection/suction velocities.

the centre of the disk. When there is injection at both the disks (V_1=0·4 and V_2=−0·2), the Nusselt number is more near the centre of the disk, than in the earlier case (V_1=0·4 and V_2=0·2) and then decreases sharply as we move away from the centre of the disk. When there is suction at both the disks, the Nusselt number is throughout less than in the earlier two cases and again a decrease is observed with increase in the distance from the centre of the disk.

The variation on Nusselt number at the upper disk against $\lambda(=r/r_0)$ has been shown in figure 3. For injection at the lower and suction at the upper disk (V_1=0·4 and V_2=0·2) the Nusselt number is throughout negative and a continuous slow increase is observed. For injection at both the disks (V_1 = 0·4 and V_2=−0·2) the Nusselt number is throughout positive and once again increases with λ. For injection at both the disks (V_1=−0·4 and V_2=0·2) the Nusselt number is negative for λ=0, but changes sign and an increase is observed with λ.

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Magnetohydrodynamic flow of a rarefied gas near an accelerated porous plate

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Abstract. An investigation is made of the flow of an electrically conducting rarefied gas due to the time-varying motion of an infinite porous plate, the gas being permeated by a transverse magnetic field. The suction is taken to be a constant and the magnetic lines of force are taken to be fixed relative to the fluid. The effects of magnetic field, rarefaction parameter, suction parameter are shown by means of some tables. The expressions of the skin friction for the two particular cases have also been obtained.

Keywords. Magnetohydrodynamic flow; rarefied gas; accelerated porous plate.

1. Introduction

Rossow [7] investigated the flow of an electrically conducting viscous liquid past an impulsively moving infinite plate in the presence of an external magnetic field. Kakutani [3] and Ong and Nicholls [6] respectively extended the problem to the case of an oscillating infinite plate with and without the induced magnetic field by the current. Similar problems in case of time-varying accelerated plate have been investigated by Gupta [2], Soundalgekar [8] and Mohapatra [5]. In all these problems, the flow of the normal density fluids has been considered. In the present era of high altitude flights, the study of rarefied gas has been recognised to be of immense importance. In the case of rarefied gases, the ordinary continuum approach fails to yield satisfactory results. When the gas is only slightly rarefied, results agreeing with the observed physical phenomena can be analysed by solving the usual Navier-Stokes equations together with modified boundary conditions allowing for a velocity slip and temperature jump at the surface. This scheme of theoretical investigation of the flow so called 'slip flow regime' is particularly suitable for studying the effects of gas rarefaction on any classical viscous flow problem. Schaaf has studied the flow of a viscous incompressible fluid past an impulsively moving infinite flat plate in slip flow regime and has been published in the book by Eckert and Drake [1]. The same problem under the influence of a magnetic field has been studied by Soundalgekar *et al* [9]. Mishra *et al* [4] has studied the flow of an electrically conducting rarefied gas due to time-varying motion of an infinite plate under the influence of a magnetic field.

In this paper, we study the flow of an electrically conducting rarefied gas past an infinite accelerated porous plate in the presence of a uniform transverse magnetic

field due to the time-varying motion of an infinite plate. General expressions of the velocity and skin friction have been obtained. It is assumed that the suction is constant and the induced magnetic field produced by the motion of the electrically conducting rarefied gas is negligible and no applied or polarization voltage exists. The skin friction in two cases namely (i) when the plate moves with the velocity Ae^{at} and (ii) when the plate moves with the velocity $Ae^{at}t$ have been found; if a (real number) is zero then it is impulsive start in the first case and it is uniformly accelerated start in the second case.

2. Formulation of the problem

A coordinate system is chosen with infinite stretched horizontal plate lying along the positive x-axis and the y-axis being perpendicular to it. A uniform and constant magnetic field H_0 is imposed along y-axis and the plate is taken as electrically non-conducting. It is assumed that the plate is uniformly porous and the magnetic permeability μ_e is constant throughout the field.

The equation of motion is

$$\frac{\partial \bar{\mathbf{V}}}{\partial t} + (\bar{\mathbf{V}} \cdot \nabla)\,\bar{\mathbf{V}} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{\mathbf{V}} + \frac{1}{\rho}\bar{\mathbf{J}} \times \bar{\mathbf{B}} \tag{1}$$

where $\bar{\mathbf{V}}$ is the velocity of the fluid, p is the pressure, ρ the density, ν the kinematic viscosity and $\bar{\mathbf{B}} = \mu_e \bar{\mathbf{H}}$, $\bar{\mathbf{B}}$ being the electromagnetic induction and $\bar{\mathbf{H}}$ the magnetic field.

The equation of continuity is

$$\nabla \cdot \bar{\mathbf{V}} = 0 \tag{2}$$

As the plate is infinite in length, all variables in this problem are functions of y and t only.

Equations of motion for unsteady flow of viscous incompressible electrically conducting rarefied gas in the presence of transverse magnetic field, leaving induced magnetic field, are

$$\frac{\partial u}{\partial t} - v_0 \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} u \tag{3}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4}$$

$$\frac{\partial v}{\partial y} = 0 \tag{5}$$

where $B_0 = \mu_e H_0$
with the boundary conditions

$$u = 0 \text{ everywhere for } t \leqslant 0 \tag{6}$$

$$\left.\begin{aligned} u &= A\, t^n e^{at} + L_1 \frac{\partial u}{\partial y} \text{ at } y = 0 \\ u &= 0 \text{ as } y \to \infty \end{aligned}\right\} \; t > 0 \tag{7}$$

where $L_1 = (2 - f_1) / f_1] L$; L being mean free path given by $[(\pi/8)^{1/2} / 0{\cdot}499]\mu$ $[(RT)^{1/2}/p]$; f_1 being the Maxwell's reflexion co-efficient and R, gas constant.

3. Solution of equations

We shall solve the problem by using the technique of Laplace transform which is defined as

$$\bar{u}(y, p) = \int_0^\infty \exp(-pt)\, u(y, t)\, dt \quad (p>0) \tag{8}$$

and its inverse

$$u(y, t) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} \bar{u}(y, p) \exp(pt)\, dp. \tag{9}$$

Multiplying (3) by $\exp(-pt)$ and then integrating between the limits 0 to ∞ and on using the condition (6), we get

$$(d^2\bar{u}/dy^2) + (v_0/\nu)(d\bar{u}/dy) - [(m+p)\bar{u}]/\nu = 0 \tag{10}$$

where $m = \sigma B_0^2/\rho$.

The boundary conditions (7) are transformed to:

$$\left.\begin{aligned} \bar{u} &= \frac{A\,(n+1)^{1/2}}{(p-a)^{n+1}} + L_1\, d\bar{u}/dy \qquad \text{at } y = 0 \\ \bar{u} &= 0 \text{ as } y \to \infty \end{aligned}\right\} \tag{11}$$

Solution of eq. (10) satisfying the conditions (11) is

$$= \bar{u} \frac{A\,\Gamma(n+1) \exp\left[-\frac{1}{2}\left\{\frac{v_0}{\nu} + \left(\frac{v_0^2}{\nu^2} + \frac{4(m+p)}{\nu}\right)^{1/2}\right\} y\right]}{(p-a)^{n+1}\left[1 + \frac{L_1}{2}\left\{\frac{v_0}{\nu} + \left(\frac{v_0^2}{\nu^2} + \frac{4(m+p)}{\nu}\right)^{1/2}\right\}\right]}. \tag{12}$$

The inverse Laplace transformation yields to

$$u(y, t) = \frac{A\,\Gamma(n+1) \exp(-v_0 y/2\nu)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp\left[pt - \frac{1}{2}\left(\frac{v_0^2}{\nu^2} + \frac{4(m+p)}{\nu}\right)^{1/2} y\right] dp}{(p-a)^{n+1}\left[1 + \frac{L_1}{2}\left\{\frac{v_0}{\nu} + \left(\frac{v_0^2}{\nu^2} + \frac{4(m+p)}{\nu}\right)^{1/2}\right\}\right]} \tag{13}$$

Put

$$(v_0^2/4v) + m = m_1,\ m_1 + p = z^2,\ b = y/\sqrt{v},\ b' = L_1/\sqrt{v},\ \alpha = m_1 + a,$$

$$b'_1 = [1 + L_1\ (v_0/2v)]/b' \tag{14}$$

Equation (13) reduces to

$$u(y, t) = [A\ \Gamma\ (n+1)/b']\ \exp\ [-(v_0 b/2\sqrt{v})]\ I(y, t, \alpha, n) \tag{15}$$

where

$$I(y, t, \alpha, n) = \frac{1}{2\pi i} \int_{Br_3} \frac{\exp\ [(z^2 - m_1)\ t - bz]\ 2zdz}{(z^2 - \alpha)^{n+1}\ (z + b'_1)} \tag{16}$$

Br_3 being the Bromwich path.
Now we find the values of $I(y, t, \alpha, n)$ for different integral values of n.
Substituting $n=0$ in (16), we get

$$I(y, t, \alpha, 0) = \frac{1}{2\pi i} \int_{Br_3} \frac{\exp\ [(z^2 - m_1)\ t - bz]\ 2zdz}{(z^2 - \alpha)\ (z + b'_1)}$$

$$= \frac{b'_1}{(\alpha - b'^2_1)} \exp\ (b'^2_1\ t + bb'_1 - m_1 t)\ \text{erfc}\ (b + 2tb'_1)/2\sqrt{t}$$

$$+ [\exp\ (at + b\sqrt{\alpha})/2\ (b'_1 - \sqrt{\alpha})]\ \text{erfc}\ (b + 2t\sqrt{\alpha})/2\sqrt{t} \tag{17}$$

$$+ [\exp\ (at + b\sqrt{\alpha})/2\ (b'_1 + \sqrt{\alpha})]\ \text{erfc}\ (b - 2t\sqrt{\alpha})/2\sqrt{t}.$$

Using (16), we have

$$nI(y, t, \alpha, n) = \frac{d}{d\alpha} I(y, t, \alpha, n-1) + t\ I(y, t, \alpha, n-1) \text{ for } n \geqslant 1. \tag{18}$$

From (17) and (18), we have

$$I(y, t, \alpha, 1) = -\frac{b'_1}{(\alpha - b'^2_1)^2} \exp\ (b'^2_1\ t + bb'_1 - m_1 t)\ \text{erfc}\ \frac{b + 2b'_1 t}{2\sqrt{t}}$$

$$+ \exp\ (at + b\sqrt{\alpha})\ \text{erfc}\ \frac{b + 2t\sqrt{\alpha}}{2\sqrt{t}}$$

$$\times \left[\frac{b}{4\sqrt{\alpha}\ (b'_1 - \sqrt{\alpha})} + \frac{1}{4\sqrt{\alpha}\ (b'_1 - \sqrt{\alpha})^2} + \frac{t}{2(b'_1 - \sqrt{\alpha})}\right]$$

$$- \exp\ (at - b\sqrt{\alpha})\ \text{erfc}\ \frac{b - 2t\sqrt{\alpha}}{2\sqrt{t}}$$

$$\times\left[\frac{1}{4\sqrt{a}\,(b'_1+\sqrt{a})^2}+\frac{b}{4\sqrt{a}\,(b'_1+\sqrt{a})}-\frac{t}{2(b'_1+\sqrt{a})}\right]$$

$$-\left(\frac{t}{\pi}\right)^{1/2}\frac{\exp\,[-((b^2/4t)+m_1t)]}{b'^2_1-a}. \tag{19}$$

Similarly we can find $I(y, t, a, n)$ for different integral values of n.
The shear stress at the plate is

$$\tau_0=\mu\,(\partial u/\partial y)_{y=0}$$

$$=-\frac{\mu A\,\Gamma(n+1)}{\sqrt{\nu}\,b'}\left[I_0(t, a, n)+\left(\frac{v_0}{2\sqrt{\nu}}-b'_1\right)I(0, t, a, n)\right] \tag{20}$$

where

$$I_0(t, a, n)=\frac{1}{2\pi i}\int_{Br_3}\frac{\exp[(z^2-m_1)t]2zdz}{(z^2-a)^{n+1}}. \tag{21}$$

Putting $n=0$ in (21), we have

$$I_0(t, a, 0)=\frac{1}{2\pi i}\int_{Br_3}\frac{\exp\,[(z^2-m_1)t]\;2zdz}{(z^2-a)}=\exp\,(at) \tag{22}$$

From (21), we have

$$nI_0\,(t, a, n)=\frac{dI_0}{da}(t, a, n-1)+tI_0(t, a, n-1)\text{ for }n\geqslant 1. \tag{23}$$

Using (17), (18), (22) and (23), we can evaluate both the integrals involved in (20) for different integral values of n.

4. Particular cases

4.1. *When the plate starts moving with velocity* $A\,\exp\,(at)$

Using (17), (20) and (22), the non-dimensional skin friction at the plate is

$$\tau_0=\frac{\left(1+h_1\frac{v_0^*}{2}\right)}{h_1\left[\left(1+h_1\frac{v_0^*}{2}\right)^2-h_1^2\,(a_0^2+at)\right]}\exp\left[\frac{1}{h_1^2}\left(1+\frac{v_0^*\,h_1}{2}\right)^2-a_0^2\right]$$

$$\times\,\mathrm{erfc}\left(\frac{1}{h_1}+\frac{v_0^*}{2}\right)+\exp\,(at)\,\mathrm{erfc}\,(a_0^2+at)^{1/2}\frac{(a_0^2+at)^{1/2}}{\left(1+h_1\frac{v_0^*}{2}\right)^2-h_1^2(a_0^2+at)}$$

$$+\exp(at)\,\frac{h_1(a_0^2+at)-\frac{v_0^*}{2}\left(1+h_1\frac{v_0^*}{2}\right)}{h_1^2(a_0^2+at)-\left(1+h_1\frac{v_0^*}{2}\right)^2} \tag{24}$$

where

$$h_1=(L_1/\sqrt{\nu t})=(b'/\sqrt{t})$$

$$\sqrt{mt}=m_0,\ v_0^*=v_0\sqrt{t/\nu} \tag{25}$$

$$a_0=[\,m_0^2+(v_0^{*2}/4)\,]^{1/2},$$

$$y^*=b/\sqrt{t}$$

4.2. *When the plate starts moving with velocity At* exp (at)

Using (19), (20), (22) and (23), the non-dimensional skin friction at the plate is

$$\tau_0=\exp(at)\left[\frac{\frac{v_0^*}{2}\left(1+h_1\frac{v_0^*}{2}\right)-h_1(a_0^2+at)}{\left(1+h_1\frac{v_0^*}{2}\right)^2-h_1^2(a_0^2+at)}\right.$$

$$\left.-\frac{h_1\left(1+h_1\frac{v_0^*}{2}\right)}{\left\{\left(1+h_1\frac{v_0^*}{2}\right)^2-h_1^2(a_0^2+at)\right\}^2}\right]+\exp(at)\ \operatorname{erfc}(a_0^2+at)^{1/2}$$

$$\times\left[\frac{(a_0^2+at)^{1/2}}{\left(1+h_1\frac{v_0^*}{2}\right)^2-h_1^2(a_0^2+at)}+\frac{1}{2(a_0^2+at)^{1/2}}\right.$$

$$\left.\times\frac{\left(1+h_1\frac{v_0^*}{2}\right)^2+h_1^2(a_0^2+at)}{\left\{\left(1+h_1\frac{v_0^*}{2}\right)^2-h_1^2(a_0^2+at)\right\}^2}\right]$$

$$+h_1\left(1+h_1\frac{v_0^*}{2}\right)\frac{\exp\left[\frac{1}{h_1^2}\left(1+h_1\frac{v_0^*}{2}\right)^2-a_0^2\right]}{\left\{h_1^2(a_0^2+at)-\left(1+h_1\frac{v_0^*}{2}\right)^2\right\}^2}$$

$$\times\operatorname{erfc}\left(\frac{1}{h_1}+\frac{v_0^*}{2}\right)+\frac{1}{\sqrt{\pi}}\,\frac{\exp(-a_0^2)}{\left(1+h_1\frac{v_0^*}{2}\right)^2-h_1^2(a_0^2+at)}\,. \tag{26}$$

Table 1. Values of skin friction ($n=0$, $at=1$)

m_0			h_1	v_0*
0·00	0·25	0·50		
1·70972	1·71781	1·74116	1	
2·48531	2·50327	2·55517	0·5	1
4·05227	4·14569	4·41112	0	
1·57826	1·58839	1·61847	1	
2·20388	2·22598	2·29007	0·5	0·5
3·07549	3·17735	3·46612	0	
2·29069	2·39592	2·69313	0	0

Table 2. Values of skin-friction ($n=0$, $m_0^2=0{\cdot}75$)

at			h_1	v_0*
0·0	0·25	0·50		
0·61614	0·80760	1·05607	1	
0·88614	1·174011	1·54981	0·5	1
1·3427	1·91412	2·67539	0	
0·56398	0·74553	0·98173	1	
0·77578	1·04099	1·38898	0·5	0·5
0·96896	1·45277	2·10231	0	
0·67491	1·08205	1·63346	0	0

5. Numerical work and conclusions

Tables 1 and 2 show the effects of the magnetic field strength m_0, rarefaction parameter h_1, suction parameter v_0* and at on the skin friction. We observe that for fixed values of (i) at, h_1 and v_0*, skin friction increases with increase in m_0 (ii) v_0*, at and m_0, it increases as h_1 decreases (iii) at, m_0 and h_1, skin friction decreases as v_0* decreases (iv) m_0, h_1 and v_0*, it increases as at increases.

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Steady flow and heat transfer in a visco-elastic fluid between two coaxial rotating disks

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Abstract. Flow and heat transfer of an elastico-viscous liquid between two parallel uniformly porous disks rotating about a common axis, are studied by perturbation technique. It is seen that for such a fluid, drag increases with suction and decreases with injection, while the rate of heat transfer increases in the presence of either suction or injection.

Keywords. Visco-elastic; flow; disks; perturbation technique.

1. Introduction

The flow in a viscous incompressible fluid between two parallel disks of infinite radii one or both of which are rotating, has been studied by a number of authors, for example by Batchelor[1], Stewartson [8], Grohne [2], Mellor *et al* [5] Narayana and Rudraiah [6], Katyal [4] and Gaur [3]. Since the flow of a fluid of elastico-viscous Walter B' model is of immense practical interest we have considered the steady flow generated in such a fluid due to the uniform rotation of a disk in the presence of another disk which may be at rest or rotating about the same axis as the first. One of the disks is in general considered to be porous. The interesting phenomenon of heat transfer usually associated with such flows has also been considered. The equations governing the motion being non-linear, we have perturbed the flow, pressure and temperature functions in terms of the fluid parameter λ, where λ is assumed to be small ($\lambda \ll 1$). The solution of the first two sets have been obtained by Runge-Kutta-Gill numerical integration technique as given in Ralston and Wilf [7].

2. Formulation

We consider two coaxial disks of infinite radii in the planes $z=0$ and $z=d$ rotating with uniform angular velocities ω and $s\omega$ about the z axis which passes through their centres, s being a real constant. It is proposed to study the flow and heat transfer in a visco-elastic liquid of Walters [9] B' model between the disks due to their rotation. The constitutive equation of the fluid are:

$$p^{ik} = -p\delta^{ik} + 2\mu g^{ik} - 2\mu_c g^{-ik}$$

$$g^{-ik} = g^{ik}_{,k}\, v^k - g^{ik}\, v^j_{,k} - g^{kj}\, v^i_{,k} + g^{ij}\, v^k_{,k} \qquad (1)$$

where p^{ik} is the stress tensor, g^{ik} is the rate of strain tensor given by,

$$2g^{ik} = v^i_{,k} + v^k_{,i}$$

v^i being a velocity vector, p the pressure and μ, μ_c two material constants. We assume the presence of suction (or injection) at the lower disk (in the plane $z=0$) by supposing it to be porous. We represent the radial, azimuthal and axial components of the fluid velocity by u, v and w, pressure and temperature at any point in the fluid by p and T, the constant temperatures of the lower and upper disks by T_0 and T_1. If the suction (or injection) velocity is taken to be w_0 and the upper disk is assumed to rotate at a constant time the speed of the lower one, the boundary conditions can be written as

$$z = 0\colon\ u = 0,\ v = r\omega,\ w = w_0,\ p = \bar{p},\ T = T_0$$

$$z = d\colon\ u = 0,\ v = sr\omega,\ w = 0,\ T = T_1. \tag{2}$$

The transformations

$$\eta = z/d,\ u = r\omega F(\eta),\ v = r\omega G(\eta),\ w = d\omega H(\eta)$$

$$p - \bar{p} = \mu\omega\{p(\eta) + (r^2/2d^2)\,k_0\},\ T - T_1 = (T_0 - T_1)\ Q(\eta)$$

reduce the equations of motion (without any external force), continuity, energy (in a cylindrical polar, (r, θ, z) system coordinates where azimuthal variations are neglected on account of axial symmetry) and the boundary conditions to

$$F'' = \lambda(HF''' + 4FF'') + R(HF' - G^2 + F^2) + K_0$$

$$G'' = \lambda(H \cdot G''' - 6F'G') + R(2FG + HG') - 2\lambda TFG$$

$$H' = -2F$$

$$P' = -2F' + 2\lambda(HF'' + 6FF') + 2RFH$$

$$Q'' = \sigma R(HQ') \tag{3}$$

and $$F(0) = 0,\ G(0) = 1,\ H(0) = \Lambda,\ P(0) = 0,\ Q(0) = 1$$

$$F(1) = 0,\ G(1) = s,\ H(1) = 0,\ Q(1) = 0 \tag{4}$$

where a dash denotes differentiation with respect to η,

$$\lambda = \mu_c\omega/\mu,\ R = d^2\omega/\nu,\ T = (d/r)^2,\ \Lambda = w_0/d\omega \text{ (porosity parameter)},$$

$\sigma = \mu c/k$ (Prandtl number), and k_0 is a constant.

3. Solution

As the viscoelasticity deviates from the Newtonian behaviour slightly we shall be justified in general to represent the velocity, pressure and temperature functions as

$$S_i = \sum_{n=0}^{\infty} \lambda^n \, [S_i(\eta)]_n \tag{5}$$

where S_i corresponds to the functions F, G, H, P and Q as i takes the values 1, 2, 3, 4 and 5 respectively. For small values of $\lambda(\lambda \ll 1)$ we neglect terms involving squares and higher powers of λ as these terms being small compared to the first two terms, influence only little the character of flow and heat transfer. Thus retaining the first two terms on the right hand side in each member in (5) and inserting them in (3) and (4) we obtain, on equating separately to zero the coefficient functions of λ and the terms free from γ, the following sets of differential equations followed by their appropriate boundary conditions, where the constant k_0 appearing in the first equations of (3) is eliminated through differentiation once with respect to η.

First set

$$\begin{aligned}
F_0''' &= R(H_0F_0'' - 2G_0G_0') \\
G_0'' &= R(2F_0G_0 + H_0G_0') \\
H_0' &= -2F_0 \\
P_0' &= -2F_0' + 2RF_0H_0 \\
Q_0'' &= \sigma R \, H_0 \, Q_0'.
\end{aligned} \tag{6}$$

Boundary conditions:

$$\begin{aligned}
&F_0(0) = 0, \; G_0(0) = 1, \; H_0(0) = \Lambda, \; P_0(0) = 0, \; Q_0(0) = 1 \\
&F_0(1) = 0, \; G_0(1) = s, \; H_0(1) = 0, \; Q_0(1) = 0
\end{aligned} \tag{7}$$

Second set

$$\begin{aligned}
F_1''' &= R(H_1F_0'' + H_0F_1'' - 2G_0G_1' - 2G_1G_0') + 4F_0'F_0'' \\
&\quad + R(2F_0 + RH_0^2)(H_0F_0'' - 2G_0G_0') - 2RH_0(F_0F_0'' + G_0'^2) \\
&\quad - 2R^2H_0G_0(2F_0G_0 + H_0G_0') \\
G_1'' &= 2RH_0G_0F_0' + R^2H_0^2(2F_0G_0 + H_0G_0') - 6F_0'G_0' \\
&\quad + R(2F_0G_1 + 2F_1G_0 + H_0G_1' + H_1G_0') - 2RTF_0G_0 \\
H_1' &= -2F_1 \\
P_1' &= -2F_1' + 2H_0F_0'' + 12F_0F_0' + 2R(F_0H_1 + F_1H_0) \\
Q_1'' &= \sigma R(H_0Q_1' + H_1Q_0').
\end{aligned} \tag{8}$$

Boundary conditions

$$F_1(0) = 0,\ G_1(0) = 0,\ H_1(0) = 0,\ P_1(0) = 0,\ Q_1(0) = 0$$

$$F_1(1) = 0,\ G_1(1) = 0,\ H_1(1) = 0,\qquad Q_1(1) = 0. \tag{9}$$

The first set which corresponds to the steady flow and heat transfer of a Newtonian fluid about uniformly rotating porous disks is seen to be a non-linear set of ordinary differential equations. The second set is however linear and its solution functions really give us a measure of the viscoelastic effect. These two sets have been numerically integrated with the step length as 0·1 on an IBM 1130 for various values of the parameters involved. The numerical results obtained have been exhibited through graphs and tables on the basis of which conclusions have been arrived at in the following. We have restricted our discussion to the case of a stationary upper disk ($s=0$).

4. Discussion

The disk in the plane $\eta=0$ is termed as lower disk and the flow or heat transfer components increase with R indicates with an increase in either the distance between the disks or the rotational speed of the disk.

4.1. *Radial flow*

For a Newtonian fluid the radial flow is outward for all layers for which $\eta<0{\cdot}47$ and is inward when $\eta>0{\cdot}47$. For a viscoelastic fluid the region of inward radial flow

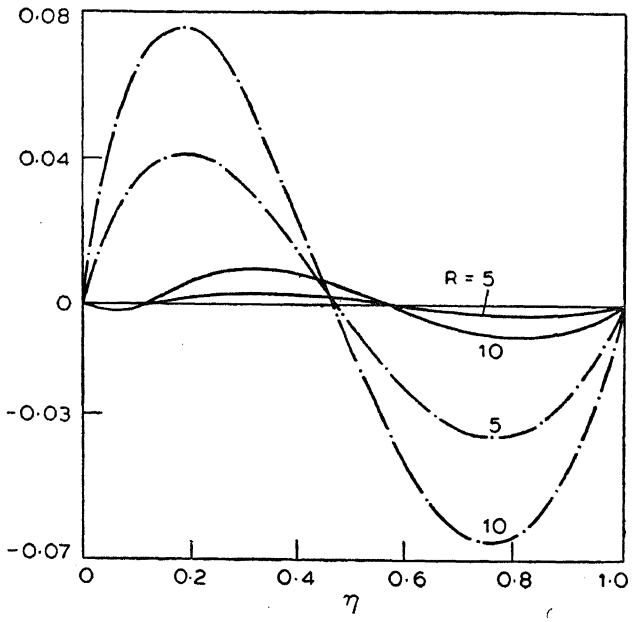

Figure 1. Radial flow functions, S=0, Λ=0, T=0·05
—·—·— F_0, ———— F_1

extends beyond $\eta=0{\cdot}47$. The effect of viscoelasticity is also in general to increase the inward or outward radial flow (as the case may be) in comparison to the Newtonian one, except only in layers adjacent to the lower disk and the central region. The viscoelastic effect on the radial flow seems to increase with an increase in R (figure 1). In the presence of a small injection ($\Lambda=0{\cdot}5$) through the lower disk any variation due to changes in R is almost counterbalanced by the effect of injection whether it is a Newtonian or a viscoelastic fluid of the model discussed here. Radial flow is much more pronounced in the presence of suction or injection. In all cases it is directed away from the disk in the presence of injection and towards the disk in the presence of suction. In the presence of a small injection velocity ($\Lambda=0{\cdot}5$) the outward radial flow is maximum midway between the disks ($\eta=0{\cdot}5$) where the viscoelastic effect is totally absent (figures 2, 3).

4.2. *Tangential flow*

For a Newtonian fluid tangential flow is observed to decrease linearly with η when $R=5$. With an increase in R, it decreases everywhere in the fluid. The viscoelasticity

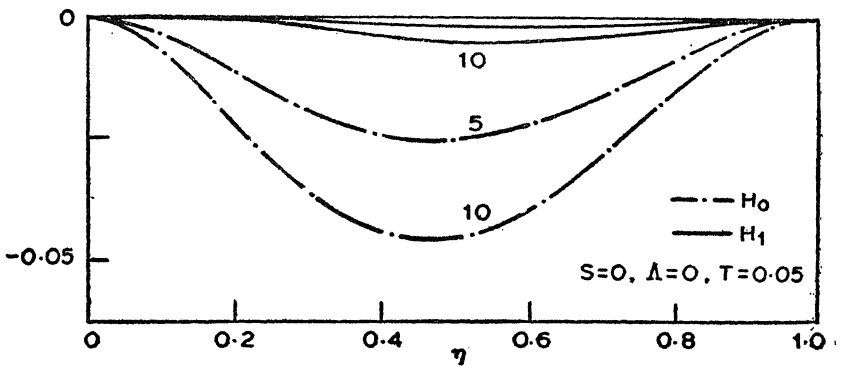

Figure 2. Axial flow functions

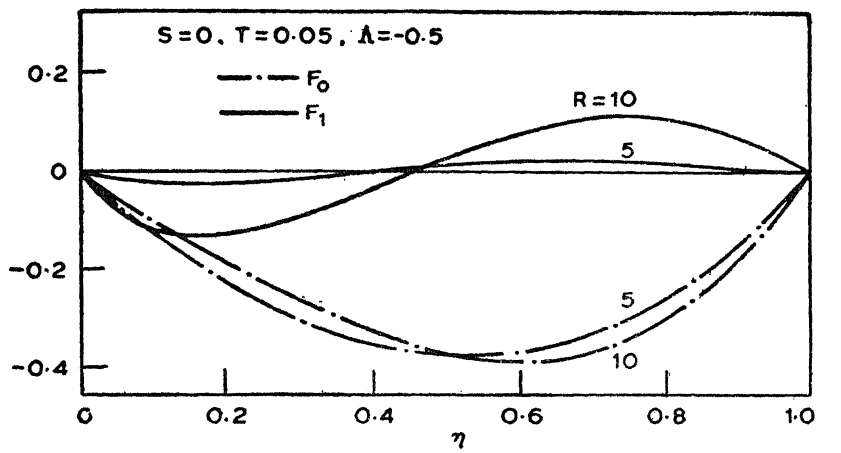

Figure 3. Radial flow functions

Table 1. Tangential flow functions ($T=0{\cdot}05$, $S=0$)

		$\Lambda = -0{\cdot}5$		$\Lambda = 0$		$\Lambda = 0{\cdot}5$	
R	η	G_0	$-G_1$	G_0	G_1	G_0	$-G_1$
	0·0	1·00000	0·00000	1·00000	0·00000	1·00000	0·00000
	0·1	0·91009	0·04055	0·89042	−0·00474	0·90585	0·18613
	0·2	0·82949	0·06432	0·78382	0·01303	0·79978	0·28646
	0·4	0·66135	0·11745	0·58126	0·05526	0·57917	0·27177
5	0·6	0·46294	0·19185	0·38694	0·06184	0·37158	0·10971
	0·8	0·23722	0·20470	0·19379	0·03127	0·18182	−0·02671
	0·9	0·11907	0·14114	0·09689	0·01238	0·09063	−0·04121
	1·0	0·00000	0·00000	0·00000	0·00000	0·00000	0·00000

in the fluid is observed to increase the tangential flow in the entire fluid region. For a viscoelastic fluid the general effect of both suction as well as injection is to decelerate the tangential velocity. While in the presence of suction the maximum deceleration occurs nearer the upper disk, in the presence of injection it occurs nearer the lower one. Also very near the upper disk the influence of injection is not to decelerate but to accelerate the rotation. However in the case of the non-porous rotating disks, viscoelasticity helps to accelerate the flow except in regions adjacent to the rotating disk (table 1).

4.3. *Axial flow*

It is directed everywhere towards the disk in the Newtonian as well as the viscoelastic fluid, the viscoelastic effect is of course very feeble especially near the disk but increases with an increase in R. The maximum influence of viscoelasticity in

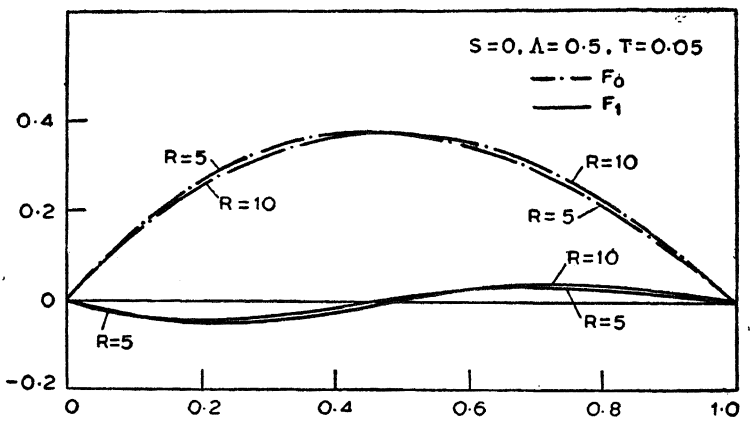

Figure 4. Radial flow functions

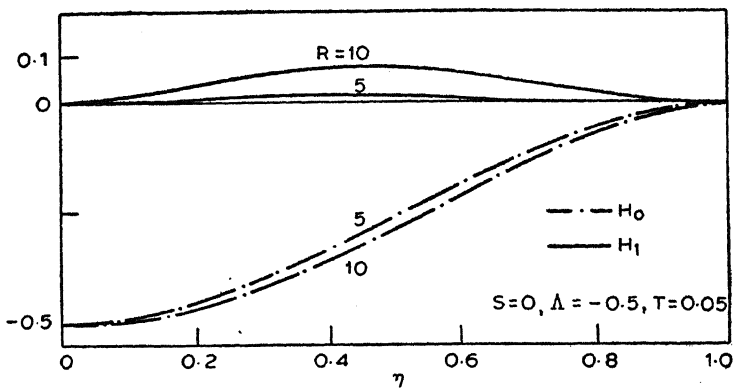

Figure 5. Axial flow functions

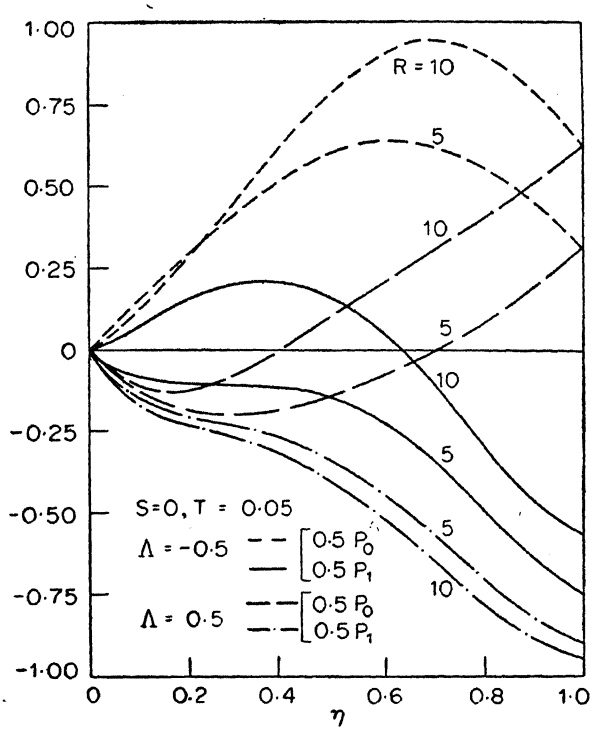

Figure 6. Pressure functions

axial flow is observed at $\eta=0{\cdot}6$ (figure 4). Presence of suction at the non-stationary disk makes the axial flow directed away from the disk for the viscoelastic fluid (figure 5).

4.4. *Pressure function*

It is seen that the pressure function in the Newtonian fluid is negative throughout the lower half and positive in the upper half of the fluid region, and that the function is evenly distributed. Viscoelastic influence on the pressure distribution is very small.

In the presence of suction at the rotating disk the difference of pressure at any point in the fluid over that at the disk always remains positive for a viscous fluid. The influence of viscoelasticity in general is to decrease the value of pressure, the decrease being more predominant towards the stationary disk. The pressure function is seen to increase with an increase in R. With injection at the rotating disk a fall of pressure in the fluid below that at the disk is observed in the region $0<\eta<0{\cdot}7$ for $R=5$ and the region $0<\eta<0{\cdot}4$ for $R=10$ (figure 6).

4.5. *Temperature function*

In a Newtonian fluid temperature gradually falls from that of the rotating one to the level of the stationary upper disk. The effect of viscoelasticity in the temperature profile is almost negligible. Presence of suction at the rotational disk has the effect of lowering down the temperature at any point in the Newtonian fluid. An increase in this temperature is observed for the viscoelastic fluid (figure 7).

4.6. *Tangential drag*

The effect of viscoelasticity is in general to increase the drag on the rotating disk except perhaps when R is small and there is suction through the rotating disk. An increase in R lowers down the drag on the rotating disk in both Newtonians as well as the viscoelastic fluid if the same disk is porous, the influence being marked in the presence of suction. If however the disk is non-porous an increase in R helps to increase the drag. Porosity in general depresses the value of the drag at the rotating disk (table 2).

Drag on the stationary disk is observed to decrease with viscoelasticity in case the rotating disk is porous, but otherwise it increases although very slightly. In the presence of suction through the rotating disk an increase in R increases the drag at the stationary disk, but in all other cases, drag decreases with increasing R. It is also seen that drag at the non-stationary disk registers an upward trend with suction at the

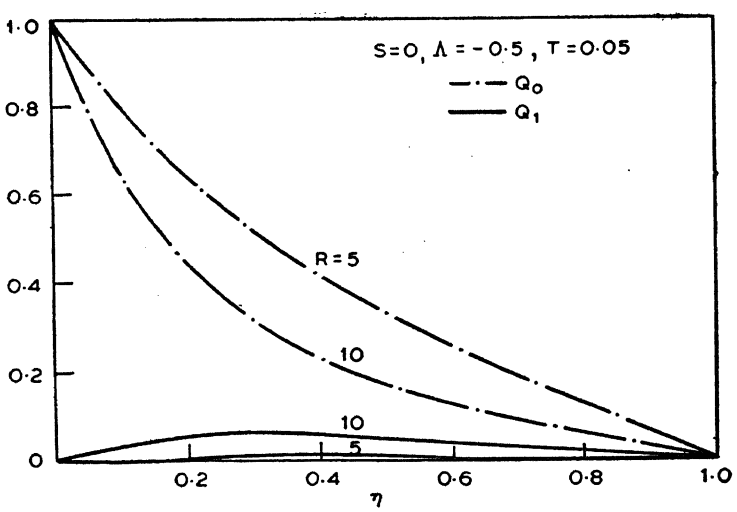

Figure 7. Temperature functions

Table 2. Tangential drags and Nusselt numbers at the disks ($T=0{\cdot}05$, $\lambda=0{\cdot}01$, $S=0$, $\sigma=1$)

		Tangential drags at the disks				Nusselt numbers at the disks			
		plane $\eta=0$		plane $\eta=1$		plane $\eta=1$		plane $\eta=1$	
Λ	R	Newtonian	Non-Newtonian	Newtonian	Non-Newtonian	Newtonian	Non-Newtonian	Newtonian	Non-Newtonian
$-0{\cdot}5$	5	$-0{\cdot}99330$	$-0{\cdot}98634$	$-1{\cdot}19154$	$-1{\cdot}15804$	$-2{\cdot}19939$	$-2{\cdot}20074$	$-0{\cdot}62651$	$-0{\cdot}62663$
	10	$-0{\cdot}14755$	$-0{\cdot}17532$	$-1{\cdot}68350$	$-1{\cdot}61008$	$-3{\cdot}38484$	$-3{\cdot}37909$	$-0{\cdot}29335$	$-0{\cdot}29239$
0	5	$-1{\cdot}10242$	$-1{\cdot}10453$	$-0{\cdot}97021$	$-0{\cdot}97093$	$-1{\cdot}00393$	$-1{\cdot}00392$	$-0{\cdot}96773$	$-0{\cdot}96770$
	10	$-1{\cdot}36371$	$-1{\cdot}37003$	$-0{\cdot}90012$	$-0{\cdot}90093$	$-1{\cdot}01559$	$-1{\cdot}01557$	$-0{\cdot}88661$	$-0{\cdot}88648$
$0{\cdot}5$	5	$-0{\cdot}85185$	$-0{\cdot}96399$	$-0{\cdot}90526$	$-0{\cdot}90121$	$-0{\cdot}38842$	$-0{\cdot}38839$	$-1{\cdot}35517$	$-1{\cdot}35545$
	10	$-0{\cdot}71580$	$-0{\cdot}71816$	$-0{\cdot}84209$	$-0{\cdot}82718$	$-0{\cdot}13914$	$-0{\cdot}13903$	$-1{\cdot}64974$	$-1{\cdot}65040$

rotating disk and a downward trend with injection, irrespective of the fluid being Newtonian or viscoelastic (table 2).

4.7. *Nusselt number*

There is no significant change in the value of the Nusselt number at either of the disks due to the presence of viscoelasticity in the fluid. It is however observed that suction at the rotating disk helps to increase its value and injection to decrease it at the porous disk. An opposite trend is noted at the stationary disk (table 2).

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Convective flow and heat transfer of a viscous heat generating fluid in the presence of a moving, infinite, vertical, porous plate

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Abstract. The analysis of convective flow and heat transfer of a viscous heat generating fluid past a uniformly moving, infinite, vertical, porous plate has been made systematically with a view to throw adequate light on the effects of the plate-motion and the presence of heat generation/absorption on the flow and heat transfer characteristics. The equations of conservation of momentum and energy which govern the flow and heat transfer of the said problem have been solved numerically by the method of Runge-Kutta-Gill. The numerical results thus obtained for the flow and heat transfer characteristics have revealed many an interesting behaviour, of the skin friction and the rate of heat transfer coefficient at the plate.

Keywords. Convective flow; fluid dynamics; heat transfer; laminar flow; viscous flow.

1. Introduction

The asymptotic suction-flow problem past an infinite flat plate was the simplest example by means of which Griffith and Meredith [4] showed the effectiveness of the suction to prevent boundary-layer separation. Taking the free stream oscillations into account Stuart [11] analysed the forced flow and heat transfer from an infinite porous plate. Lighthill [5] made a systematic study of the effect of the free stream oscillations on the laminar skin friction and heat transfer. Soon after the advent of this pioneering work there has been a host of papers in the literature using Lighthill's technique. Comparatively of recent origin has been the work of Soundalgekar [7, 8]. Certain errors that have entered in the paper of Soundalgekar [7] and have threatened its quality have been brought to light and corrected by Vajravelu and Sastri [14]. Soundalgekar and Gupta [9] have extended the papers of the former author taking into account the steady motion of the porous plate.

In the literature there are examples wherein the important role played by the heat sources constant or variable in the analysis of flow and heat transfer of a heat-generating fluid has been given adequate attention. Recently convection studies with internal heat sources have been proposed for Earth's mantle (Gaskell [3], Runcorn [6] Tozer [12]) and for the outer region of star interiors (Bethe [1]). Engineering processes in which a fluid supports an exothermic chemical or nuclear reaction are very common today and the correct process design requires accurate correlations for the heat transfer coefficients at the boundary surfaces. Despite its increasing importance in the technological and physical problems

natural convection studies with heat generation has received relatively little treatment. With a view to supply heat transfer data in this area a systematic attempt has been made in this paper to analyse the steady flow and heat transfer of a viscous heat generating fluid in the presence of a steadily moving, infinite, vertical, porous plate taking into account the temperature-dependent volumetric heat generation/absorption, the occurrence of which in physical problems have been pointed out by Foraboschi and Federico [2] and Sparrow and Cess [10]. The title problem for the unsteady case will be reported in a separate paper. In § 2 the equations which govern the flow and heat transfer of the problem subject to the relevant boundary conditions have been presented and reduced to a system of ordinary differential equations which have been integrated numerically by the method of Runge-Kutta-Gill. A discussion of the numerical results of the flow and heat transfer characteristics is given in § 3, which is followed up by a summary of the important results in § 4.

2. Formulation and solution of the problem

The steady incompressible flow of a viscous heat generating fluid past an infinite, vertical, uniformly moving porous plate with constant suction is assumed to be two-dimensional. Along the length of the plate upwards is the x'-axis and normal to it y'-axis. The flow being incompressible all the fluid properties are taken to be constant except the density in the buoyancy-force term where the Boussinesq approximation is used. With the foregoing assumptions and closely following Soundalgekar [7] the equations which govern the flow and heat transfer of the problem became

$$-v_0(\partial u'/\partial y') = \nu(\partial^2 u'/\partial y'^2) + g_x\beta\,(T'-T'_\infty), \tag{1}$$

$$-v_0(\partial T'/\partial y') = (K/\rho' C_p)\,(\partial^2 T'/\partial y'^2) + (\nu/C_p)\,(\partial u'/\partial y')^2 + (Q/\rho' C_p)(T'_\infty - T'). \tag{2}$$

The last term in (2) denotes the heat generation/absorption varying directly with the temperature difference and the symbols have their usual meanings. The pertinent boundary conditions are

$$u' = V', T' = T'_w \quad \text{at } y' = 0,$$
$$u' = U_\infty, T' = T'_\infty \quad \text{as} \quad y' \to \infty. \tag{3}$$

Setting the nondimensional variables

$$y = y'v_0/\nu,\ u = u'/U_\infty \text{ and } \theta = (T'-T'_\infty)/(T'_w - T'_\infty)$$

in (1) and (2) and in the boundary conditions (3) we get

$$(d^2u/dy^2) + (du/dy) + G\,\theta = 0, \tag{4}$$

$$(d^2\theta/dy^2) + P(d\theta/dy) + PE\,(du/dy)^2 - \alpha\,\theta = 0 \tag{5}$$

and

$$u = V, \quad \theta = 1 \quad \text{at } y = 0,$$

$$u = 1, \quad \theta = 0 \quad \text{as } y \to \infty. \tag{6}$$

where $G = \nu g_x \beta (T'_w - T'_\infty)/U_\infty v_0^2$, modified Grashof number,

$E = U_\infty^2/C_p (T'_w - T'_\infty)$, Eckert number,

$P = \mu C_p/K$, Prandtl number,

$\alpha = Q\nu^2/Kv_0^2$, heat source parameter,

$V = V'/U_\infty$, nondimensional plate velocity.

The second-order ordinary differential equations (4) and (5) subject to the boundary conditions (6) are solved by the well known Runge-Kutta-Gill method as used in Vajravelu [13]. Giving several sets of values to the nondimensional parameters G, E, P, α and V, we have calculated the values of the nondimensional velocity u, nondimensional temperature θ, the wall skin friction τ $(= \tau'/\rho' U_\infty v_0 = (du/dy)_{y=0})$ and the rate of heat transfer coefficient at the plate Nu $(=(\mathrm{Nu}'\nu/Kv_0)(T'_w - T'_\infty) = (d\theta/dy)_{y=0})$ numerically on EC 1030 and are presented in figures 1 to 8. While giving positive (negative) values to the free convection parameter G $(\propto (T'_w - T'_\infty))$ and the frictional heating parameter E $(\propto (T'_w - T'_\infty))$ we observe that originally cool (hot) fluid gets heated (cooled) subsequently.

3. Discussion of the results

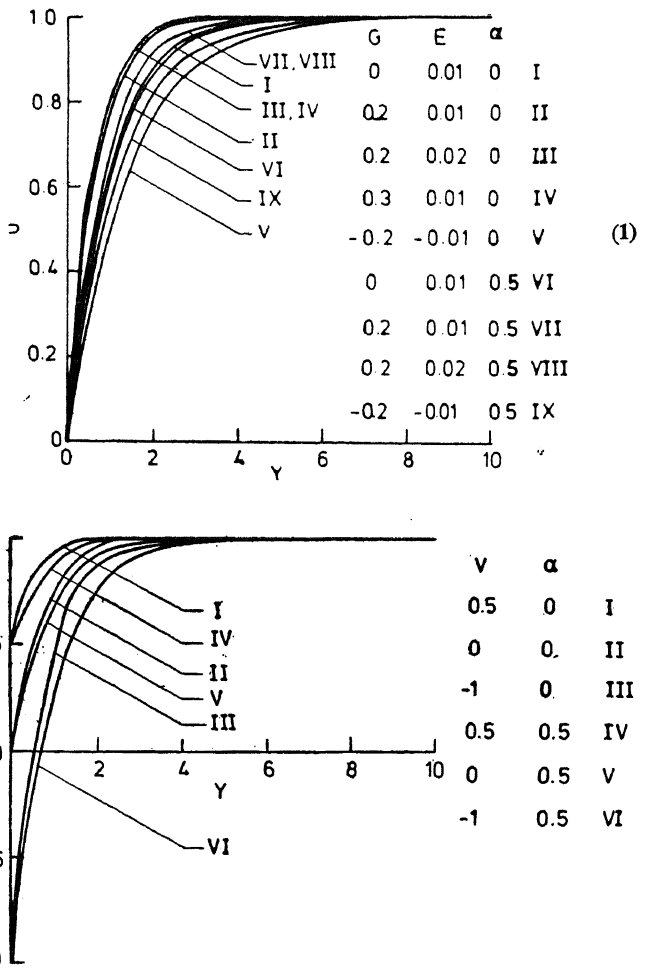
At the outset it is worth mentioning that when G takes small values, physically it is equivalent to saying that either the viscosity is small or the temperature differences are small; in the former case the Prandtl number of the fluid is taken small. On the other hand, if the temperature differences are assumed to be small, the Prandtl number of the fluid can be moderately large. Therefore we restrict our attention to fluid-flows with small temperature—differences and moderate Prandtl numbers. Also when α takes positive values, the heat generation/absorption term—$\alpha\theta$ in (5) implies that there is heat generation in the flow whenever $T'_w < T'_\infty$ and heat absorption whenever $T'_w > T'_\infty$. Therefore by the α-term in (5) is meant heat generation some times and heat absorption at other times. However, in what follows, α is called the heat source parameter only.

3.1. *Velocity*

Figure 1 describes the flow pattern of air ($P = 0{\cdot}71$) with changes in the free convection parameter G, the frictional heating parameter E and the heat source parameter α in the case when the plate is stationary. For the fluid-heating case $(T'_w > T'_\infty)$ the velocity increases with corresponding increases in the parameters G and E and it

diminishes in the presence of heat sources. On the other hand, for the fluid-cooling case ($T'_w < T'_\infty$), the effect of heat sources is always to increase the velocity of the fluid. Again, when the plate moves in a direction same as or opposite to that of the free stream the effect of the parameters G and E on the fluid velocity are qualitatively similar to those in the stationary plate case and this is true whenever $T'_w \gtrless T'_\infty$.

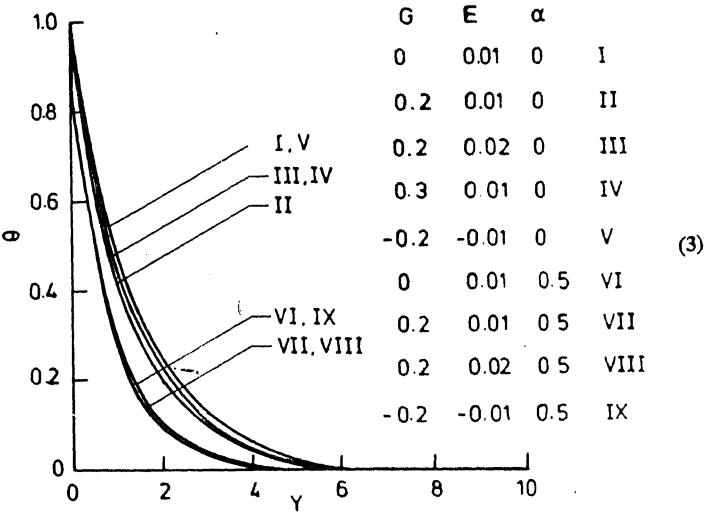
From figure 2 it is noticed that the fluid (air) velocity increases gradually when the plate moves rapidly in the direction of the free stream and decreases slowly when it moves opposite to that of the free stream. In both the cases however, the heat source decreases the fluid velocities considerably. Again the fluid velocities diminish considerably in the presence of heat sources for the fluid-heating case and increases for the fluid-cooling case. This phenomenon holds good both when the plate is at rest and when it is in motion.



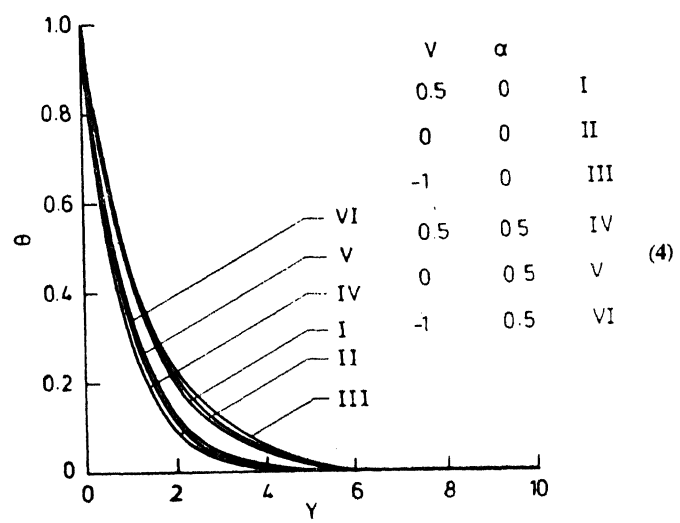
Figures 1 and 2. Velocity profiles. **1.** $P=0{\cdot}71$ and $v=0$. **2.** $G=0{\cdot}2$, $E=0{\cdot}01$ and $P=0{\cdot}71$.

3.2. *Temperature*

For the stationary plate case, temperatures of air diminished in the presence of heat sources and this holds good both in the fluid-heating case and in the fluid-cooling case (see figure 3). However in the fluid-heating case, an increase in G and E results in a corresponding increase in the temperature. Also the observation that the temperatures are diminished in the presence of heat sources is valid both when the plate is at rest and when it is in motion. The above mentioned result is true in the fluid-heating case as well as in the fluid-cooling case. However the effects of the parameters G and E on the temperature in the moving plate case are qualitatively the same as in the stationary plate case.



(3)



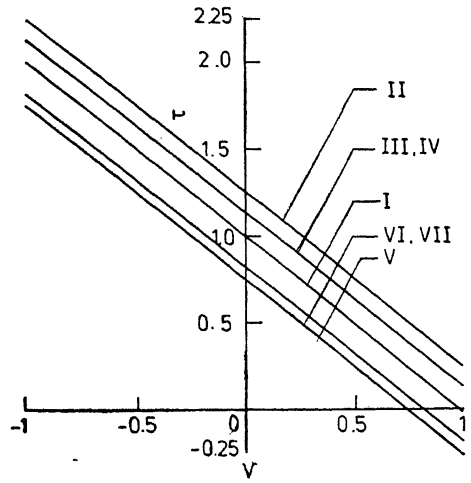
(4)

Figures 3 and 4. Temperature profiles. **3.** $P=0{\cdot}71$ and $\nu=0$. **4.** $G=0{\cdot}2$, $E=0{\cdot}01$ and $P=0{\cdot}71$.

From figure 4 it is observed that when the fluid is heated up the temperature decreases in the presence of heat sources. The fluid temperature diminishes also when the plate moves in the direction of the free stream while it decreases considerably when the plate moves opposite to it. This increasing (decreasing) behaviour of temperature is mainly due to the fact that the velocity derivative and hence the frictional work increases (decreases) when the plate moves in a direction opposite to (same as) that of the free stream.

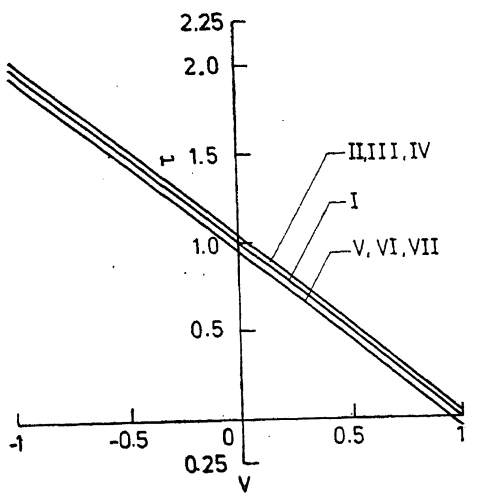
3.3. *Skin friction*

In the case of air (P=0·71) the skin friction at the plate decreases as the plate velocity increases but increases with the free convection parameter G. However the skin friction is hardly affected by the frictional heating. Again in the presence of heat sources the skin friction diminishes for the fluid-cooling case (see figure 5).



G	E	α	
0	0.01	0	I
0.2	0.01	0	II
0.2	0.01	0.5	III
0.2	0.02	0.5	IV
-0.2	-0.01	0	V
-0.2	-0.01	0.5	VI
-0.2	-0.02	0.5	VII

(5)



G	E	α	
0	0.01	0	I
0.2	0.01	0	II
0.2	0.01	0.5	III
0.2	0.02	0.5	IV
-0.2	-0.01	0	V
-0.2	-0.01	0.5	VI
-0.2	-0.02	0.5	VII

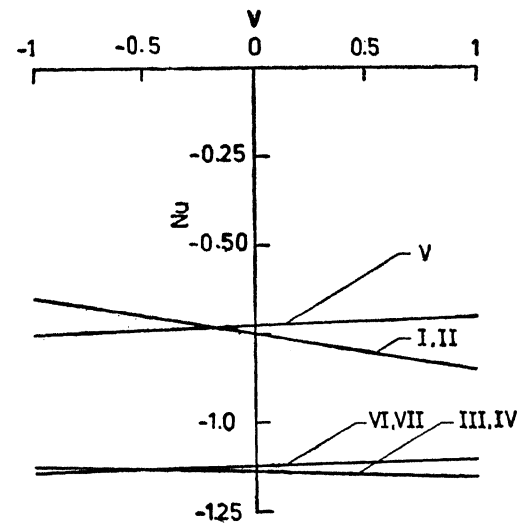
(6)

Figures 5 and 6. Skin friction. **5.** P=0·71. **6.** P=7.

As in the case of air here in the case of water ($P=7$) also the effect of the parameter G is to increase the skin friction while the effects of the other parameters E and α are found to be negligible. However as in the case of air here too, the skin friction is a decreasing function of the plate velocity (see figure 6). A close look at figures 5 and 6 reveal an important result that the skin friction becomes zero when the plate velocity attains a critical value equal to 0·75 (approximately) in the case of air and 0·95 (approximately) in the case of water and this happens in the fluid-cooling case only. It means physically that at around the critical plate-velocity, there are indications for reverse flow to exist.

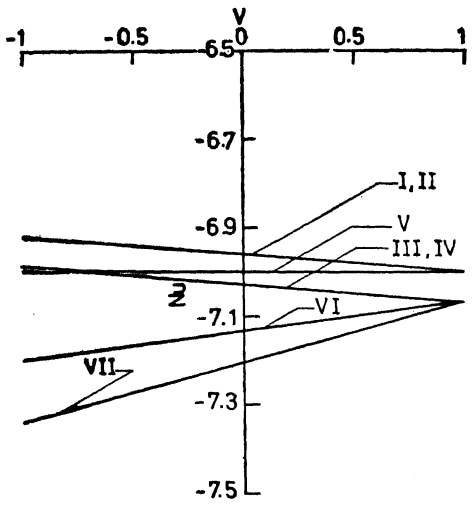
3.4. *Rate of heat transfer coefficient at the plate*

In both air and water, it is observed that the rate of heat transfer coefficient at the plate treated as a function of plate-velocity is a decreasing function in the



G	E	α	
0	0.01	0	I
0.2	0.01	0	II
0.2	0.01	0.5	III
0.2	0.02	0.5	IV
-0.2	-0.01	0	V
-0.2	-0.01	0.5	VI
-0.2	-0.02	0.5	VII

(7)



G	E	α	
0	0.01	0	I
0.2	0.01	0	II
0.2	0.01	0.5	III
0.2	0.02	0.5	IV
-0.2	-0.01	0	V
-0.2	-0.01	0.5	VI
-0.2	-0.02	0.5	VII

(8)

Figures 7 and 8. Rate of heat transfer. **7.** $P=0·71$. **8.** $P=7$.

fluid-heating case and an increasing one in the fluid-cooling case. This increase in the rate of heat transfer coefficient at the plate with the plate-velocity is not so prominent in the presence of heat sources as in their absence. In both the cases of water and air, the heat source diminishes the rate of plate-heat transfer coefficient considerably and this is true both in the fluid-heating case and in the fluid-cooling case. However the effects of the parameters G and E on the heat transfer coefficient at the plate are insignificant. Finally it is observed that the values of the rate of heat transfer coefficient at the plate are always negative indicating physically that there is always heat flow from the plate to the fluid only (see figures 7 and 8).

4. Conclusions

(i) It has been found that there are chances for the fluid-velocity to exceed the free stream velocity although the plate-velocity is smaller than the free stream. (ii) For the moving plate case, the presence of heat sources diminish the velocity in the fluid-heating case and enhances in the fluid-cooling case. (iii) The effect of increasing Prandtl number has been found to diminish the fluid-velocity as well as the fluid-temperature. (iv) The fluid-temperatures always decrease in the presence of heat sources and increase considerably with an increase in either the free convection parameter G or the frictional heating parameter E. This is true both when the plate is at rest and when it is in uniform motion. (v) There is a marked reduction in the values of the fluid-temperature when the plate moves in the direction of the free stream and the reverse is the case when it moves opposite to the free stream. (vi) The skin friction has been found to be a decreasing function of the plate-velocity in the case of water as well as in the case of air. (vii) In the fluid-cooling case the skin friction has been noticed to change its sign from positive to negative attaining the zero value when the plate-velocity is about 0·75 in the case of air and 0 9 in the case of water. This indicates physically that there are chances for the reverse flow regimes to exist. (viii) Like the skin friction, the rate of heat transfer coefficient at the plate too is a decreasing function of the plate-velocity and it is always negative indicating physically that there is heat flow only from the plate to the fluid. (ix) The presence of heat sources has been found to diminish the rate of heat transfer coefficient at the plate considerably and this is as much true in the case of water as in the case of air.

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Point force in a stratified fluid in a cylinder under a magnetic field

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Abstract. The flow induced by a body moving in an inviscid incompressible density stratified fluid in an infinite circular cylinder under the influence of a uniform axial magnetic field is studied using the method of replacing the body by an isolated point force. This method was adopted by Childress and others in discussing the body effects in a viscous fluid. The solution is obtained using the Fourier transformation and the Lighthill's radiation condition. The cases of weak and strong magnetic fields are discussed.

Keywords. Point force; stratified fluid; radiation condition.

1. Introduction

When an axisymmetric body moves along its axis of revolution in an unbounded density stratified fluid with or without rigid rotation, linearised equations fail to throw any light on the steady features of the flow since the ultimate motion continues to be unsteady either in the whole region or certain parts of it. This led a few authors (Krishna [4, 5]) to investigate how the flow behaves in a bounded or partially bounded medium. It was established that for a disturbance created by the motion of the disk in a stratified fluid with or without rigid rotation, bounded partially by an infinite cylinder, the steady state linearised equations yield a unique solution. In fact, the solution was determined by allowing waves in the downstream side and using matching conditions across the disk. It was observed that these waves in the leeside of the boundary are infinite in number and are persistent. Battaiah [1] studied the effect of a uniform transverse magnetic field on a rotating fluid using a similar configuration. In constrast to the non-magnetic case, the leewaves in the downstream side are finite in number and decay at sufficiently far distances.

In discussing the general nature of such flow patterns induced by a body moving through a fluid either in an unbounded or partially bounded medium, the method of replacing a body by a suitable singularity distribution or by an external impulse in proper direction has attracted the attention of several authors in recent times, notably Childress [2], Lai [6] and Janowitz [3]. Lai used a suitable singularity distribution to obtain the flow of a rotating inviscid fluid past a nearly spherical obstacle in a pipe. Childress studied the slow motion of a sphere in a rotating viscous fluid replacing the sphere by an isolated point force in terms of delta functions. Janowitz discussed the waves in a stratified fluid taking into account the body effects through a discontinuous momentum integral which in turn gives rise to an isolated external impulse along the direction of motion of the body.

Following the same technique, we discuss the body effects in an inviscid, density stratified, electrically conducting fluid of large conductivity confined in a circular cylinder under the influence of a uniform magnetic field along the axis of the cylinder. The steady Oseen type MHD linearised equations under Boussinesq approximation, taking into account the induced magnetic field, are solved to yield a unique solution.

2. Governing equations and solution

Choosing the cylindrical polar frame (r, θ, z), we consider an incompressible, inviscid, density stratified and electrically conducting fluid moving with a uniform velocity U far upstream, confined in a circular cylinder radius a with its axis in the z-direction, in the presence of a uniform magnetic field H_0 along the axis. Perturbations are created in the fluid region by the action of an isolated point force of strength $-F_0$ situated at the origin. The steady Oseen type linearised equations of motion for an axisymmetric flow, under Boussinesq approximation are

$$\rho_0' U(\partial u/\partial z) = -(\partial p/\partial r) + \mu_e H_0 [(\partial h_r/\partial z) - (\partial h_z/\partial r)] \tag{1}$$

$$\rho_0' U(\partial v/\partial z) = \mu_e H_0 (\partial h_\theta/\partial z), \tag{2}$$

$$\rho_0' [(1/2\pi r) F_0 \delta(r/a) \delta(z/a) + U(\partial w/\partial z) = -(\partial p/\partial z) - \rho g, \tag{3}$$

$$U(\partial h_r/\partial z) = H_0(\partial u/\partial z), \tag{4}$$

$$U(\partial h_\theta/\partial z) = H_0(\partial v/\partial z), \tag{5}$$

$$U(\partial h_z/\partial z) = H_0(\partial w/\partial z), \tag{6}$$

$$U(\partial \rho/\partial z) + w(\partial \rho_0/\partial z) = 0, \tag{7}$$

$$(\partial u/\partial r) + (u/r) + (\partial w/\partial z) = 0, \tag{8}$$

$$\rho_0 = \rho'_0(1 - \beta z) \tag{9}$$

where $(u, v, W+w)$ are the components of the velocity, (h_r, h_θ, H_0+h_z) are the components of the magnetic field, ρ the density, p the pressure and μ_e the magnetic permeability. Using the equation of continuity (8), we define ψ such that

$$u = (1/r)(\partial\psi/\partial z), \; w = -(1/r)(\partial\psi/\partial r). \tag{10}$$

We define the following non-dimensional variables:

$$\psi' = (\psi/Ua^2), \; r' = (r/a), \; z' = (z/a), \; u' = (u/U), \; w' = (w/U)$$

$$p' = (p/\rho U^2), \; h'_r = (h_r/H_0), \; h_\theta' = (h_\theta/H_0), \; h'_z = (h_z/H_0), \; \rho = (U^2\rho'_0\rho'/ga).$$

Eliminating p from (1) to (3) and using (4) to (10), the governing equation in ψ, in the nondimensional form, on dropping the affix is

$$(1-k)\,(\partial^2/\partial z^2)\,[(\partial^2\psi/\partial z^2)+(\partial^2\psi/\partial r^2)-(1/r)\,(\partial\psi/\partial r)]+F^2\,[(\partial^2\psi/\partial r^2)$$

$$-(1/r)\,(\partial\psi/\partial r)] = (F_0/2\pi U^2)\,(\partial/\partial z)\,(\delta(z))\,[(\partial/\partial r)\,(\partial(r))-(\partial(r)/r)] \quad (11)$$

where

$F = (\beta g a^2/U^2)^{1/2}$ (Modified Froude number) and

$k = \mu_e H_0^2/\rho'_0 U^2$ (Magnetic pressure number).

The boundary conditions to be satisfied are

$$(\partial\psi/\partial z) = 0 \text{ on the cylinder } r = 1 \quad (12)$$

$$\psi \to 0 \text{ as } z \to \pm\infty. \quad (13)$$

We define the Fourier transform of $\psi\,(r, z)$ as

$$\bar{\psi}\,(r, \alpha) = \int_{-\infty}^{+\infty} \psi(r, z) \exp(-i\alpha z)\, dz. \quad (14)$$

On using (13) and (14), (11) reduces to

$$\frac{\partial^2\bar{\psi}}{\partial r^2} - \frac{1}{r}\frac{\partial\bar{\psi}}{\partial r} - \lambda^2\bar{\psi} = M\,[(\delta(r)/r) - (\partial/\partial r)\,(\delta(r))] \quad (15)$$

where

$M = i\,F_0\,\alpha/2\pi U^2\,[(1-k)\,\alpha^2 - F^2]$ and

$\lambda^2 = (1-k)\,\alpha^4/[(1-k)\,\alpha^2 - F^2]$.

The boundary conditions on the cylinder in terms of $\bar{\psi}$ is

$$\bar{\psi} = 0 \text{ on } r = 1. \quad (16)$$

The solution of (15) satisfying the boundary condition (16) is

$$\bar{\psi} = M\lambda r\,\frac{K_1(\lambda)\,I_1(\lambda r) - K_1(\lambda r)\,I_1(\lambda)}{I_1(\lambda)} \quad (17)$$

Using the Fourier inversion theorem, we have

$$\psi(r, z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{\psi}(r, \alpha) \exp(i\alpha z)\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{iF_0\alpha\lambda r\,[K_1(\lambda)\,I_1(\lambda r) - K_1(\lambda r)\,I_1(\lambda)]}{I_1(\lambda)\,2\pi U^2\,[\alpha^2(1-k) - F^2)]} \exp(i\alpha z)\, d\alpha. \quad (18)$$

The integrand is an even function of λ and therefore $\lambda=0$ is not a branch point. Also, the integrand has only simple poles and these are the roots of $I_1(\lambda)=0$. Thus, they are given by $\lambda^2+\beta_n^2=0$ i.e.,

$$\alpha^2 = \frac{-(k-1)\beta_n^2 \pm [(k-1)^2\beta_n^4 - 4(k-1)\beta_n^2F^2]^{1/2}}{2(k-1)}$$

where $J_1(\beta_n)=0$ for all integral values of n. From the expansion of the integrand in the neighbourhood of $\alpha=0$, it can be seen that $\alpha=0$ is not a pole. In evaluating the integral in (18), we choose the semi-circular contour which lies in the half-plane Im $\alpha > 0$ for $z > 0$ and in the halfplane Im $\alpha < 0$ for $z < 0$.

Case 1 $(k > 1)$

The poles of the integrand are at

$$\pm (a_n \pm ib_n) \text{ for } n \leqslant N,$$

$$\pm ic_n, \pm id \text{ for } n > N, \text{ where}$$

$$a_n^2,\ b_n^2 = \mp\frac{\beta_n^2}{4} + \frac{\beta_n F}{2\,[k-1]^{1/2}}$$

$$c_n^2,\ d_n^2 = \frac{(k-1)\beta_n^2 \pm [(k-1)^2\beta_n^4 - 4(k-1)\beta_n^2F^2]^{1/2}}{2(k-1)}$$

and N is the positive integer such that

$$(k-1)\beta_N^2 < 4F^2 \text{ and } (k-1)\beta_{N+1}^2 > 4F^2.$$

The poles that contribute to the downstream flow $(z > 0)$ are

$$\pm a_n + ib_n,\ ic_n,\ id_n.$$

The poles that contribute to the upstream flow $(z < 0)$ are

$$\pm a_n - ib_n,\ -ic_n,\ -id_n.$$

Using the residue theorem, we obtain

$$\psi_+ = \sum_1^N 2\,[\lambda_n(r)\cos a_n z + \mu_n(r)\sin a_n z]\exp(-b_n z)$$

$$+ \sum_{N+1}^{\infty} [-F_1(c_n^2, r)\exp(-c_n z) - F_1(d_n^2, r)\exp(-d_n z)] \qquad (z > 0) \qquad (19)$$

$$\psi_- = \sum_1^N 2\,[-\lambda_n(r)\cos a_n z + \mu_n(r)\sin a_n z]\exp(b_n z)$$

$$+ \sum_{N+1}^{\infty} [F_1(c_n^2, r)\exp(c_n z) + F_1(d_n^2, r)\exp(d_n z)] \qquad (z<0) \qquad (20)$$

where

$$\lambda_n, \mu_n = \frac{F_0 r\beta_n^2 J_1(\beta_n r)\; Y_1(\beta_n)}{4U^2(k-1)\,J_0(\beta_n)}\,[F_n D_n - E_n G_n, \; - D_n G_n - E_n F_n],$$

$$F_n = [L_n(L_n + F^2) + M_n^2]\,/\,[M_n^2 + (L_n + F^2)^2],$$

$$G_n = M_n F^2\,/\,[M_n^2 + (L_n + F^2)^2],$$

$$D_n = (a_n^2 - b_n^2)\,/\,(a_n^2 + b_n^2)^2, \; E_n = -2a_n b_n\,/\,(a_n^2 + b_n^2)^2.$$

$$L_n = (k-1)(a_n^2 - b_n^2) + F^2, \; M_n = 2a_n b_n (k-1),$$

$$F_{1,2}(\alpha^2, r) = \frac{F_0 r\beta_n^2\, Y_1(\beta_n)\, J_1(\beta_n r)\,[\pm(k-1)\,\alpha^2 - F^2]}{\pm\, 4U^2(k-1)\,\alpha^2 J_0(\beta_n)\,[\pm(k-1)\,\alpha^2 - 2F^2]}$$

Case 2. $(k < 1)$

In this case, the poles of the integrand in (18) are at $\pm\alpha_n$, $\pm i\gamma_n$ where

$$\alpha_n^2, \; \gamma_n^2 = \frac{\mp(1-k)\,\beta_n^2 + [(1-k)^2\,\beta_n^4 + 4(1-k)\,\beta_n^2 F^2]^{1/2}}{2(1-k)}$$

The simple pole $+i\gamma_n$ contributes to the downstream flow and $-i\gamma_n$ contributes to the upstream flow. The remaining poles $\pm\alpha_n$ lie on the real axis and lead to ambiguity in choosing the proper contour of integration. This ambiguity can be eliminated by considering their behaviour in the steady oscillatory problem. In the oscillatory case where the isolated point force has oscillatory strength $F_0\exp(i\omega t)$, the poles on the real axis are $\pm\alpha_n + \omega(a/U)\,\alpha'_n$ (for small ω) where $\alpha'_n = -(\alpha_n^2 + \beta_n^2)/(1-k)\,(2\alpha_n^2 + \beta_n^2)$. On using radiation condition, Lighthill [7], in order to eliminate the incoming waves (by substituting $\omega \to i\epsilon$ for ω and taking the limit $\epsilon \to 0$) these poles get shifted to the upper half-plane Im $\alpha > 0$ and contribute to the downstream flow. Now talking the limit $\omega \to 0$ (zero frequency case), we obtain the unique solution corresponding to the present case

$$\psi_+ = -\sum_1^{\infty} - F_1(\gamma_n^2, r)\exp(-\gamma_n z) + \sum_1^{\infty} - 2F_2(\alpha_n^2, r)\cos\alpha_n z \qquad (z>0) \qquad (21)$$

$$\psi_- = \sum_1^{\infty} F_1(\gamma_n^2, r)\exp(\gamma_n z) \qquad (z<0) \qquad (22)$$

The case $\beta = 0, \; k \neq 0$

The solution for $k<1$ is

$$\psi_+ = \sum_1^\infty - F_1(\beta_n^2, r) \exp(-\beta_n z) + \sum_1^\infty - 2\, G(\beta_n, r) \qquad (z>0) \qquad (23)$$

$$\psi_- = \sum_1^\infty F_1(\beta_n^2, r) \exp(\beta_n z) \qquad (z<0) \qquad (24)$$

The solution for $k>1$ is given by

$$\psi_+ = \sum_1^\infty (F_1(\beta^2{}_n, r) \exp(-\beta_n z) - G(\beta_n, r)) \qquad (z>0) \qquad (25)$$

$$\psi_- = \sum_1^\infty (F_1(\beta^2{}_n, r) \exp(\beta_n z) - G(\beta_n, r) \qquad (z<0) \qquad (26)$$

where

$$G(\beta_n, r) = - F_0\, r\; Y_1(\beta_n)\, J_1(\beta_n r)/4U^2\,(k-1)\, J_0(\beta_n).$$

3. Discussion of the results

From the expressions for ψ in (21) and (22), it can be observed that in the case $k<1$, there are infinite number of persistent leewaves in the downstream side apart from an infinite number of decaying modes. For $k>1$, both the upstream and downstream solutions consist of a finite number of oscillations which die down at far distances by an order $\exp(-b_n z)$ and an infinite number of exponentially decaying modes.

If $\qquad F/(k-1)^{\frac{1}{2}} - \beta_N/2 < 1/\beta_N,$

the influence of the oscillatory modes persist to larger distances of order

$$[\beta_N F/(k-1)^{\frac{1}{2}} - \beta_N{}^2/2]^{-1}$$

compared to the exponentially decaying modes which vanish much rapidly at distances of order β_n^{-1} $(n>N)$. If $(F/(k-1)^{\frac{1}{2}})-(\beta_1/2)<0$, the finite number of oscillations will be absent. Thus, in contrast to a weak magnetic field $(k<1)$, a strong magnetic field eliminates the appearance of an infinite number of persistent waves and induces an infinite number of exponentially decaying oscillations both in the upstream and in the downstream side. When the magnetic field is weak $(k<1)$, the persistent leewave terms in the non-homogeneous case reduce to the one-dimensional flow terms in the homogeneous case $(\beta=0)$. For $k<1$, the one-dimensional flow terms appear only in the lee side (in the limit $\beta\to 0$) whereas for $k>1$, the one-dimensional flow terms appear both in the upstream and in the downstream sides.

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Aerodynamic noise from wave-turbulence interaction

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Abstract. In order to estimate the acoustic energy scattered when a unit volume of free turbulence, such as in free jets, interacts with a plane steady sound wave, theoretical expressions are derived for two simple models of turbulence: eddy model and isotropic model. The effect of convection by mean motion of the energy-bearing eddies on the incident sound wave and on the sound generated from wave-turbulence interaction is taken into account. Finally, by means of a representative calculation, the directionality pattern and Mach number dependence of the noise so generated is discussed.

Keywords. Aeroacoustics; aerodynamic noise; wave-turbulence interaction.

1. Introduction

Aerodynamic noise is a problem of statistical fluid dynamics where statistical functions (e.g., velocity correlation) rather than basic disturbances (such as velocity) play significant role. The present investigation aims at a type of aerodynamic noise scattered from wave-turbulence interaction. As pointed out by Kraichnan [10], the problem of the scattering of sound by a turbulent flow is a most conspicuous acoustical phenomenon associated with low Mach number turbulence. In the past, the problem has been discussed in some detail by Lighthill [12], Kraichnan [10], Ford and Meecham [6] and Howe [9], among others. The analysis carried out in these references is based on a flow model in which, in the words of Lilley *et al* [13], the sound sources may move but not the flow. Therefore, the effect of convection is not recognized explicitly. In the present analysis the convection of pressure disturbance by mean motion of energy-bearing eddies is explicitly taken into account. Also, unlike the implicit form of solutions obtained in these references for the intensity of scattered energy, the present analysis provides a quantitative information so as to draw a more realistic picture of scattered energy as a function of exit Mach number, emission angle and frequency and amplitude of sound wave incident upon the flow.

2. Formulation of the problem

The analysis to be followed here is essentially a problem of interaction where two distinct fundamental oscillatory disturbances in the small-amplitude fluctuating motion of a fluid, are involved. The two disturbances are: turbulence (or, irregular

vorticity diffusion showing random variations with space and time) and sound (or acoustic wave motion). Let $u_i'(X, t)$ be the fluctuating component of the total flow velocity of the turbulent flow, and V_i be the particle velocity of the plane sound wave incident upon the flow. Then the mathematical description of pressure p of the sound generated solely by nonlinear interaction of turbulence with incident sound wave may be given in terms of an approximate version of the generalized aerodynamic noise equation as (Chandraker and Munjal [1], Munjal and Chandraker [14]).

$$\left(\frac{\partial^2}{\partial t^2} + 2U_0 \frac{\partial^2}{\partial x_1 \partial t} + U_0^2 \frac{\partial^2}{\partial x_1^2} - a_0^2 \frac{\partial^2}{\partial x_i^2}\right) \hat{p}\,(X, t) = 2\gamma \frac{\partial u_i'}{\partial x_j} \frac{\partial V_j}{\partial x_i}, \tag{1}$$

where $$\hat{p} = p/p_0 \tag{2}$$

The subscripts i and j are tensor notations. The coefficient γ is the ratio of specific heats defined by the relation

$$a_0^2 = \frac{\gamma p_0}{\rho_0} \tag{3}$$

The quantities U_0, a_0, ρ_0 and p_0 are the average values of convection velocity, ambient sound speed, ambient density and ambient pressure, respectively. X and t are, respectively, space and time co-ordinates. The x_1-axis is assumed to be aligned with the axis of the jet. Equation (1) neglects the effects of the gradients of mean flow. Further, the mean flow has been assumed to extend even beyond the source region. These approximations have been made for mathematical convenience.

The solution of (1) in an unbounded medium may be written by the method of Kirchoff's retarded potential solution technique as (Ribner [15])

$$\hat{p}\,(X, t) = \frac{\gamma}{2\pi \beta_{\text{av}} a_0^2} \int \left(\frac{\partial u_i'}{\partial y_j} \frac{\partial V_j}{\partial y_i}\right)_{(Y, t'')} \frac{d^3\, Y}{|\, X' - Y'\,|} \tag{4}$$

where

$$|\, X' - Y'\,| = \left\{\left(\frac{x_1 - y_1}{\beta_{\text{av}}}\right)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2\right\}^{1/2}$$

$$t'' = \beta_{\text{av}}\, t - \frac{|\, X' - Y'\,| - M_0\, x_1'}{a_0}$$

$$\beta_{\text{av}} = (1 - M_0^2)^{1/2}, \quad M_0 = U_0/a_0,$$

$$X' = \left(\frac{x_1}{\beta_{\text{av}}}, x_2, x_3\right), \quad Y' = \left(\frac{y_1}{\beta_{\text{av}}}, y_2, y_3\right). \tag{5}$$

X and Y are respectively observer and source co-ordinate systems. $d^3 Y$ is an infinitesimal volume element of source region. Integration is over all possible values of Y. The quantity with subscript (Y, t'') is to be evaluated at position Y and time t''.

Assuming the turbulence to be in compressible (Goldstein and Rosenbaum [8]), i.e.,

$$\partial u'_i/\partial y_i = 0, \tag{6}$$

one obtains

$$\frac{\partial u_i'}{\partial y_j}\frac{\partial V_j}{\partial y_i} = \frac{\partial^2 (u_i' V_j)}{\partial y_i \partial y_j} - u_i' \frac{\partial^2 V_j}{\partial y_j \partial y_i}. \tag{7}$$

Following Lighthill [11] and Goldstein [7] one may easily show for $|X| \gg |Y|$

$$\int \left[\frac{\partial^2 (u_i' V_j)}{\partial y_i \partial y_j}\right]_{(Y, t'')} \frac{d^3 Y}{|X' - Y'|}$$

$$\simeq \int a_i a_j \frac{\partial^2 [u_i' V_j]}{\partial t^2}(Y, t'')\frac{d^3 Y}{|X' - Y'|}, \tag{8}$$

where

$$a_i = \left(\frac{1}{a_0 \beta_{\mathrm{av}} \beta^{(i)}}\right)\left(M_0 \delta_{i1} - \frac{x_i' - y_i'}{|X' - Y'|}\right), \tag{9}$$

$$\beta^{(i)} = \beta_{\mathrm{av}} \text{ if } i = 1,$$

$$= 1 \quad \text{if } i \neq 1, \tag{10}$$

δ_{ij} is Kronecker delta. The proof of (8) is straightforward if one makes use of the divergence theorem twice in left-hand-side and then neglect resulting terms of order $(|X' - Y'|^{-2})$ and $(|X' - Y'|^{-1})$ in favour of those of the order $|X' - Y'|$.

Noting that the typical frequencies of flow-fluctuations u_i' at a point are small compared with those of the incident sound wave V_i, one obtains (Kraichnan [10], Lighthill [12])

$$\frac{\partial^2 (u_i' V_j)}{\partial t^2} \simeq u_i' \frac{\partial^2 V_j}{\partial t^2}. \tag{11}$$

An expression for particle velocity due to plane steady sound wave incident upon flow may be written as (Lighthill [12])

$$V_j = \epsilon a_0 \cos (k_c x_1 - \omega t)\, \delta_{j1}, \tag{12}$$

where

$$1 \gg \epsilon > 0, \tag{13}$$

from the requirement of small amplitude motion. ω is the radian frequency of incident sound wave. k_c is convected wave number defined by

$$k_c \equiv \omega/(a_0 + U_0) \tag{14}$$

A far-field approximation common in jet noise literature is (Lighthill [11], Goldstein [7]) (for $|X| \gg |Y|$)

$$x_i - y_i \simeq x_i, \; i = 1, \; 2, \; 3. \tag{15}$$

Generally, the approximation is valid only in the middle of emission angle range $80° > \theta > 10°$ (where θ is the emission angle measured from the jet axis in a co-ordinate system, which has its origin at the jet exit centre). The approximation $x_1 - y_1 \simeq x_1$ is not valid for $\theta > 82{\cdot}5°$. Similarly the approximation $x_2 - y_2 \simeq x_2$ (or $x_3 - y_3 \simeq x_3$) is not valid for $\theta < 7{\cdot}5°$.

A modified form of the approximation proposed by the authors in their earlier work is (Munjal and Chandraker [14], Chandraker and Munjal [3]),

$$x_i - y_i \simeq x_i - y_i^*, \tag{16}$$

where y_i^* is some hypothetical constant value of y_i chosen judiciously so as to represent the location of most intense sound sources and may be interpreted as a 'characteristic location' of all sound sources distributed in whole or part of the jet field.

Substitution of (7), (8), (11), (12) and (16) in (4) yields

$$\hat{p}\,(X, t) = \gamma\, A_i \int (u_i'\, V_1)_{(Y,\, t'')}\, d^3 Y, \tag{17}$$

where

$$A_i = \frac{-\omega^2\, a_i^*\, a_1^*}{2\pi\, \beta_{av}\, a_0^2\, |X' - Y'^*|^2} + \frac{k_c^2\, \delta_{i1}}{2\pi\, \beta_{av}\, a_0^2\, |X' - Y'^*|}, \tag{18}$$

$$a_i^* = \left(\frac{1}{a_0\, \beta_{av}\, \beta^{(i)}}\right) \left(M_0\, \delta_{i1} - \frac{x_i' - y_i'^*}{|X' - Y'^*|}\right). \tag{19}$$

3. Expression for scattered energy

Mean rate of scattered energy crossing unit area at X may be defined as (Lighthill [12], Goldstein [7]),

$$I(X) = \frac{p_0^2}{\rho_0\, a_0}\, \overline{\hat{p}^2}, \tag{20}$$

where symbol 'overbar' denotes time averaging. From (17) and (20)

$$I(X) = \frac{p_0^2\, \gamma^2\, A_0\, A_j}{\rho_0\, a_0} \int\int \overline{(V_1)_1\, (V_1)_2\, (u_i')_1\, (u_j')_2}\, d^3 Y_1\, d^3 Y_2, \tag{21}$$

where the subscripts 1 and 2 outside the parentheses refer to (Y_1, t_1'') and (Y_2, t_2''), respectively. The times t_1'' and t_2'' are defined as

$$t_1'' = \beta_{av} t + \{M_0 x_1' - |X - Y_1'|\}/a_0,$$

$$t_2'' = \beta_{av} t + \{M_0 x_1' - |X - Y_2'|\}/a_0. \tag{22}$$

On defining

$$Y_1 = Y, \; Y_2 = Y + \zeta \text{ (or } \zeta = Y_2 - Y_1), \tag{23}$$

one may write expression for mean rate of scattered energy per unit volume of turbulence and crossing unit area at X as

$$I(X/Y) = \frac{p_0^2 \gamma^2 A_i A_j}{\rho_0 a_0} \int\int \overline{(V_1)_1 (V_1)_2} \cdot \overline{(u_i')_1 (u_j')_2}\, d^3\zeta, \tag{24}$$

where

$$I(X) \equiv \int I(X/Y)\, d^3 Y.$$

The integrand in (24) is in a form of product of two time-averaged terms. The equivalence of two integrands (one of (21) and the other of (24)) is natural if one notes that the fluctuations in particle velocity due to sound wave incident upon the flow are statistically independent of the flow fluctuations (Lighthill [12]).

For the observer in the far field ($|X| \gg |Y|$) and making use of (12) and the notion of ' characteristic location ' (Munjal and Chandraker [12]) one may write

$$\overline{(V_1)_1 (V_2)_2} = \tfrac{1}{2} \epsilon^2 a_0^2 \cos(\tilde{k}_c' \cdot \zeta), \tag{25}$$

where

$$\tilde{k}_c' = (\tilde{k}_{c_1}/\beta_{av}, \tilde{k}_{c_2}, \tilde{k}_{c_3}),$$

$$\tilde{k}_{c_i} = \frac{\omega}{a_0} \frac{(x_i' - y_i'^*)}{|X' - Y'^*|}. \tag{26}$$

From the assumption of ' acoustically compact sources ', Lighthill [11, 12] shows that

$$\overline{(u_i')_1 (u_j')_2} = Q_{i,j}(Y, \zeta) \tag{27}$$

where the correlation function $Q_{i,j}$ is independent of time.

Making use of (25) and (27) in (24) one obtains

$$I(X/Y) = \frac{1}{2} \left(\frac{\epsilon^2 a_0 p_0^2 \gamma^2 A_i A_j}{\rho_0} \right) \int \cos(\tilde{k}_c', \zeta)\, Q_{i,j}(Y, \zeta)\, d^3\zeta. \tag{28}$$

The integral in (28) is evaluated here for two simple hypothetical structures of turbulence.

3.1. *Eddy model of turbulence*

It is a variant of the eddy model extensively used by Ffowcs Williams [5] and Ribner [15] among others. In this model a simple Gaussian distribution is assumed for the correlation $Q_{i,j}$ as

$$Q_{i,j}(Y,\zeta) = \overline{u_0^2}(Y)\, a_{ij} \exp\left\{-\frac{1}{2}\left(\frac{\zeta_1^2}{l_1^2}+\frac{\zeta_2^2}{l_2^2}+\frac{\zeta_3^2}{l_3^2}\right)\right\}, \tag{29}$$

where $a_{ij} = 1$ for all values of i and j.

$(\overline{u_0^2})^{1/2}$ is some characteristic turbulence velocity. l_1, l_2 and l_3 are scales of turbulence in three principal directions.

Following the standard integration formulae and making use of (3) and (29) in (25), one obtains

$$I(X/Y) = 2^{1/2}\pi^{3/2}\, a_{ij}\, \epsilon^2\, \rho_0\, a_0^5\, \overline{u_0^2}\,(Y)\, A_i\, A_j$$
$$l_1 \cdot l_2 \cdot l_3 \cdot \exp\left\{-\tfrac{1}{2}\left(\tilde{k}_{c'1}^2\, l_1^2 + \tilde{k}_{c'2}^2\, l_2^2 + \tilde{k}_{c^2 3}\, l_3^2\right)\right\}. \tag{30}$$

3.2. *Isotropic model of turbulence*

Another widely used turbulence structure is the isotropic model. This model is assumed to be locally valid for small volume of jet. The unknown statistical average values $(\overline{u_0^2})^{1/2}$ and l have been measured by Davies *et al* [4] for free jets. The analytical description of isotropic model is usually stated as follows (Goldstein [7])

$$Q_{i,j}(Y,\zeta) = \frac{\overline{u_0^2}}{2l^2}\left[\exp\left\{-\frac{1}{2l^2}(\zeta_1^2+\zeta_2^2+\zeta_3^2)\right\}\right.$$
$$\left.\left(\left\{2l^2-(\zeta_1^2+\zeta_2^2+\zeta_3^2)\right\}\delta_{ij}-\zeta_i\,\zeta_j\right)\right], \tag{31}$$

where l is the longitudinal macroscale of turbulence.

Substitution of (3) and (31) in (28) and then integration, yields

$$I(X/Y) = 2^{-1/2}\pi^{3/2}\, \epsilon^2\, \rho_0\, a_0^5\, \overline{u_0^2}\,(Y)\, A_i\, A_j\, l^5$$
$$\left\{(\tilde{k}_{c'1}^2+\tilde{k}_{c'2}^2+\tilde{k}_{c'3}^2)\,\delta_{ij}-\tilde{k}'_{ci}\,\tilde{k}'_{cj}\right\}$$
$$\exp\left\{-\frac{l^2}{2}(\tilde{k}_{c'1}^2+\tilde{k}_{c'2}^2+\tilde{k}_{c'3}^2)\right\}. \tag{32}$$

3.3. *Comparison with existing theories*

A short comparison with the existing theories may be in order here. Kraichnan [10] derived an expression in implicit form for the power spectrum $\langle p^+(n,\omega_s)\rangle$ of the scattered energy flux for isotropic turbulence structure as

$$\langle p^+(n,\omega_s)\rangle = \frac{\pi\,\omega_s^4}{8\,a_0^8}\left(\frac{\sin^2\theta}{4\sin^2\frac{\theta}{2}+\delta^2}\right).\ F. \tag{33}$$

The implicitness appears through unknown function F, the four-dimensional spectrum density of turbulence. The factor δ is defined as

$$\delta = (|\omega_s| - \omega_0)/\omega_0,$$

ω_s and ω_0 are respectively the frequency of scattered noise and the sound wave component. Vector n_i is defined as

$$n_i = x_i/|X|.$$

Lighthill [12] derived an expression for $I(X/Y)$ as

$$I(X, Y) \sim \rho_0 a_0^{-1} \epsilon^2 \omega^2 \cos^2\theta \cot^2\left(\frac{\theta}{2}\right) |X|^{-2} \cdot E, \tag{34}$$

where function E, the unknown turbulence energy spectrum, is unknown.
When one compares results (30) and (32) of the present text with (33) and (34), one finds that the present results are in explicit form enabling one to easily draw qualitative and quantitative information relevant to particular wave-scattering problem.

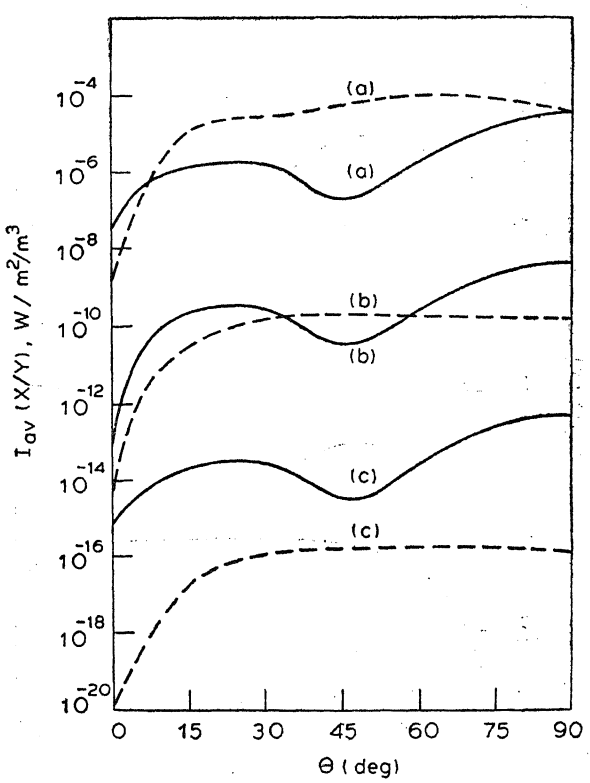

Figure 1. Directivity of noise scattered. $M_e = 0{\cdot}5$ and $\epsilon = 2{\cdot}0 \times 10^{-5}$. ——, eddy model; - - - -, isotropic model. (a) $= 10^4$ Hz, (b) $f = 10^3$ Hz, (c) $f = 10^2$ Hz.

4. A representative evaluation for a round jet

In this section an evaluation is carried out for a unit volume of turbulence located at the mid-point of the mixing region of a turbulent round jet. The central region of annular mixing part of the jet is the most intense noise producing region. Evaluation is carried out for the following set of conditions:

M_e : 0·3, 0·4, 0·5, 0·6, 0·7, 0·8, 0·9

ϵ : 2×10^{-6}, 2×10^{-5}, 2×10^{-4}

f : 10^2, 10^3, 10^4 Hz (circular frequency)

θ : 0, 15, 30, 45, 60, 75, 90 degrees

where M_e is Mach number at jet exit based on ambient sound speed and $f=\omega/2\pi$. θ is emission angle measured from the jet axis. The results for $I(X/Y)$ are expressed in watts/m^2/m^3. The input data needed to evaluate $I(X/Y)$ are taken from various sources (Goldstein [7], Davies *et al* [4], Ribner [15])

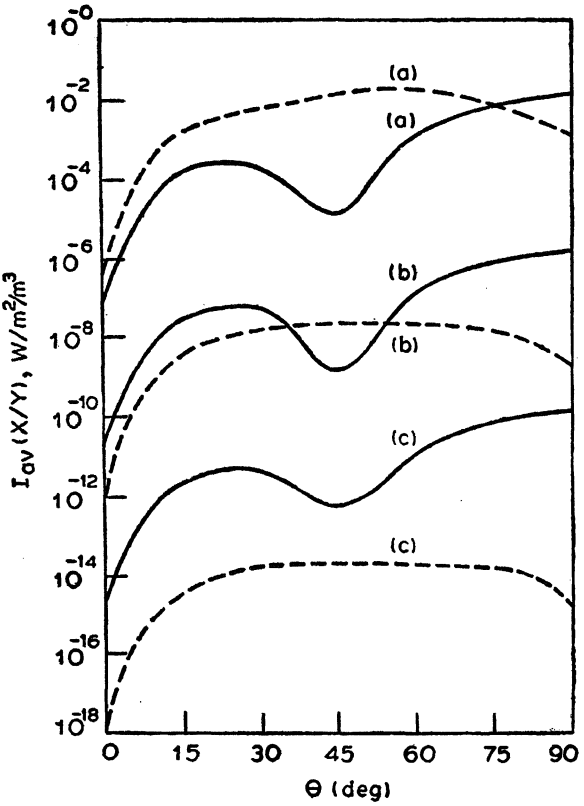

Figure 2. Directivity of noise scattered. $M_e=0{\cdot}7$ and $\epsilon = 2{\cdot}0\times10^{-5}$, ——, eddy model; - - - -, isotropic model. (a) $f=10^4$ Hz, (b) $f=10^3$ Hz, (c) $f=10^2$ Hz.

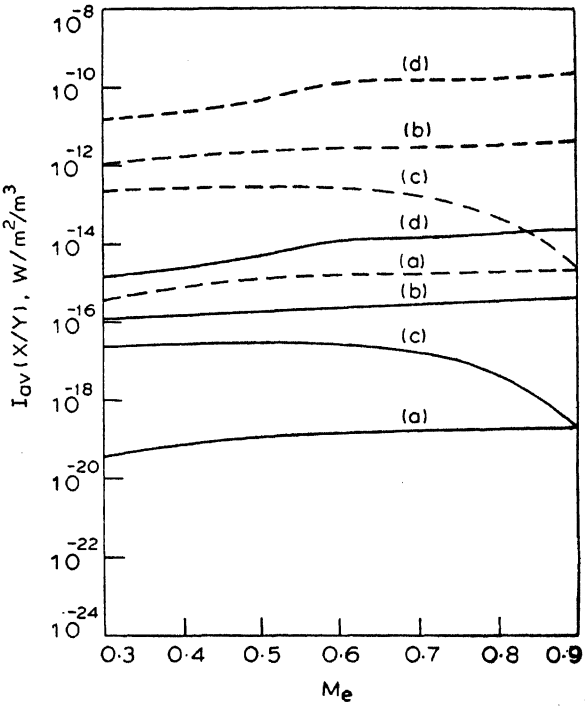

Figure 3. Mach number dependence of noise scattered. Eddy model, $\epsilon = 2{\cdot}0 \times 10^{-6}$. ———, $f = 10^2$ Hz; - - - - $f = 10^3$ Hz. (a) $\theta = 0°$, (b) $\theta = 15°$, (c) $\theta = 45°$, (d) $\theta = 90°$.

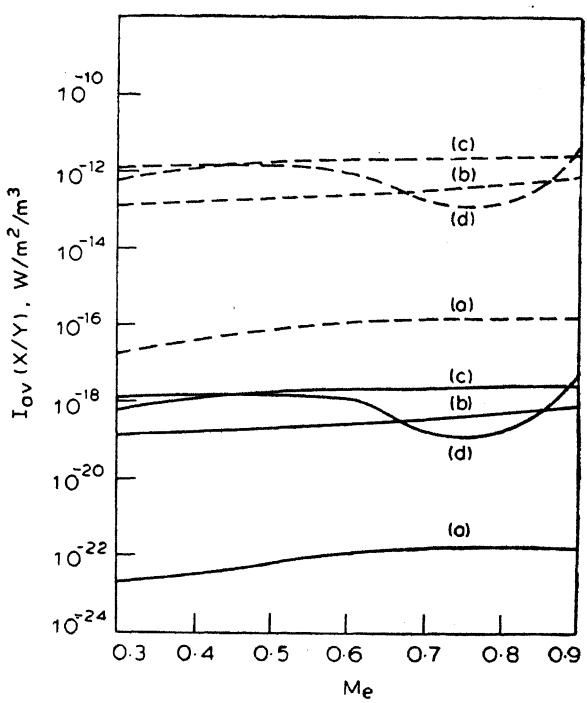

Figure 4. Mach number dependence of noice scattered. Isotropic mc $2{\cdot}0 \times 10^{-6}$, ———, $f = 10^2$ Hz; - - - - $f = 10^3$ Hz. (a) $\theta = 0°$, (b) $\theta = 15°$, (c (d) $\theta = 90°$.

as
$$y_1^* = 2D,\ (y_2^{*2} + y_3^{*2})^{1/2} = 0{\cdot}5\ D,$$
$$D = 25\ \text{mm},\ R = 3\ \text{m}$$
$$\rho_0 = 1{\cdot}177\ \text{kg/m}^3,\ a_0 = 347{\cdot}6\ \text{m/s}$$
$$(\overline{u_0^2})^{1/2} = 0{\cdot}14\ M_e a_0,\ l_1 = 0{\cdot}65\times 10^{-2}\ \text{m},$$
$$l_2 = l_3 = 0{\cdot}2\times 10^{-2}\text{m},\ l = 0{\cdot}709\times 10^{-2}\ \text{m},$$
$$d^3Y = (85{\cdot}12\times 10^{-6}\, y_1^3)\ \text{m}^3.$$

Computation is carried out for $\phi = 45°$ where ϕ is circumferential or azimuthal angle made by projection of the observer vector X in x_2-x_3 plane with the x_2-axis. It is assumed that the characteristic location of source, observer and origin, all lie in a single straight line.

Figures 1, 2, 3 and 4 show the pattern of results for the above sample data.

5. Results and conclusions

The following points emerge from the preceding sections:

(a) The present analysis is an improvement over the earlier work in that it yields explicit expressions for scattered energy.

(b) It also provides more qualitative and quantitative information on wave-scattering problem by taking into account the effect of convection by mean motion of energy-bearing eddies upon the sound incident upon the flow as well as on the sound scattered.

(c) A representative evaluation shows:

(d) (i) For isotropic model of turbulence, the energy scattered is minimum at low angles of emission ($\theta < 20°$), but remains, more or less, constant at other angles. The directivity pattern for eddy model of turbulence has a secondary minimum at or around $\theta = 45°$.

(ii) Scattered energy remains more or less constant with the change in M_e for various emission angles and sound frequencies.

Another point to be noted here concerns with flow-acoustic interaction effects in free jets. It is commonly believed that 'refraction dimple' i.e. a minimum sound level near the jet axis ($\theta < 15°$) arises due to flow-acoustic interaction effects (Goldstein [7]). As θ increases 'dimple' almost flattens out. This particular feature is observable in figures 1 and 2 where sound radiation appears to be minimum near $\theta = 0°$.

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